Complex Numbers Review

Tuesday, September 28, 2021 6:37 PM

- Converting between algebraic and polar:

Given Complex Number in algebraic form:

$$r + cj = Ae^{j\phi} \quad \text{where} \quad A = \sqrt{r^2 + c^2}, \quad \phi = \tan^{-1}\left(\frac{c}{r}\right)$$

Given Complex Number in polar form:

$$Ae^{j\phi} = r + cj \quad \text{where} \quad r = A\cos(\phi) \quad c = A\sin(\phi)$$

Note: Given $b^{j\phi}$, since $b = e^{\ln(b)}$, then $b^{j\phi} = e^{j(\ln(b))\phi}$

Note: $\cos(x) = \dfrac{e^{jx} + e^{-jx}}{2}$, $\sin(x) = \dfrac{e^{jx} - e^{-jx}}{2j}$

Signals & Systems

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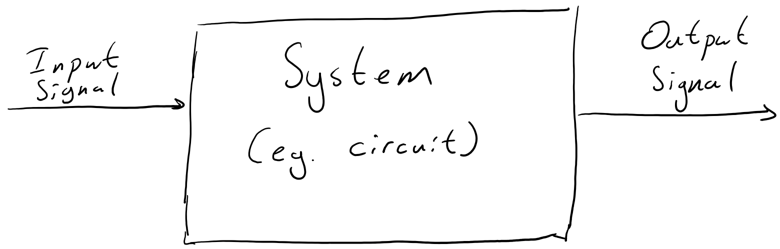
<u>Def</u> Systems are some box which takes an input signal and processes an output signal. Example: circuit.



<u>Def</u> Signals are some measurable property that is a function of time.

<u>Goal</u> Find relationships between a system and the transformation of the input signal to the output signal.

<u>Note</u> Systems considered will be <u>LINEAR</u> (eg. RLC circuits)

Linearity, Properties of RLC Components

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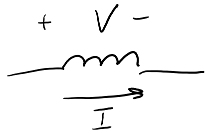
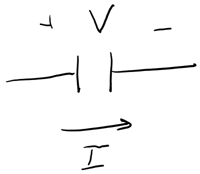
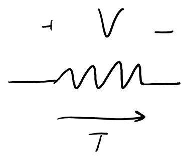
Component	Symbol	Property	Relationship	Phasor
Inductor		L - inductance	$V(t) = L \frac{\partial I(t)}{\partial t}$	$V = j\omega L I$
Capacitor		C - capicitance	$I(t) = C \frac{\partial V(t)}{\partial t}$	$V = \frac{1}{j\omega L} I$
Resistor		R - resistance	$V(t) = R I(t)$	$V = R I$

<u>Note</u> these relationships are linear because derivatives are linear operators, which leads to "linearity" of the system

<u>Def</u> if a system is linear, then any linear combination of inputs, the output will be a linear combination of each input's corresponding output.

Linear System Properties

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1) superposition principle holds

2) sinusoid in, sinusoid out; <u>the frequency stays the same</u>

Sinusoidal Functions

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Typical sinusoidal function:

$$A \cos(\omega t + \phi)$$

A - amplitude

ω - frequency

ϕ - phase

Idea Transform all quantities in a system into complex (phasor) form, then solve the circuit using complex, then convert back to real signals.

Phasor Representation:

Suppose $f(x) = A\cos(\omega t + \phi)$

then: $f(x) = A e^{j\phi}$ ← phasor form

Impedence

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All RLC relationships are in the form:

$V = Z I$, where Z is the complex number Impedence

Phasor Method

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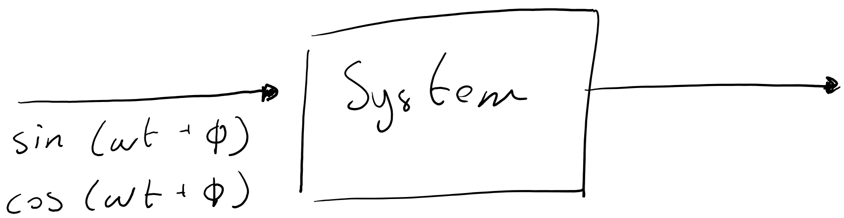
- When to use Phasor Method:



compute response to a system to sinusoidal signal

- Steps:
    1) Find phasor for the input
    2) Change all circuit components to complex impedences
    3) Solve the problem using complex algebra ($V = ZI$)
    4) Convert solution to real domain

Superposition

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- Suppose the input is the sum of multiple sinusoids:
  use superposition to solve each signal's component individually,
  the solution is the sum of each signal's interaction with the system:

Ex: Input $X(t) = Ae^{j\phi} + Be^{j\theta} + \dots$

$$\downarrow \qquad \downarrow$$

Output $Y(t) = A'e^{j\phi'} + B'e^{j\theta'} + \dots$

Frequency Response

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- Given some circuit with input V_{in}, and output V_{out}

  $H(\omega) = \frac{V_{out}}{V_{in}}$ is the frequency response

  and thus for any V_{in}:

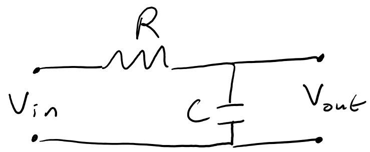
  $$V_{out} = H(\omega) \cdot V_{in}$$

  and if $V_{in} = A\cos(\omega t + \phi)$ then $V_{out} = |H(\omega)| \cdot A\cos(\omega t + \phi + \angle H(\omega))$

- Any system can be represented by a frequency response $H(\omega)$
- Any system can be represented as a filter

- Low Pass Filter:

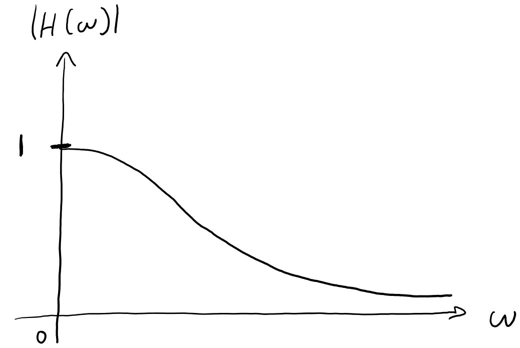
Given:

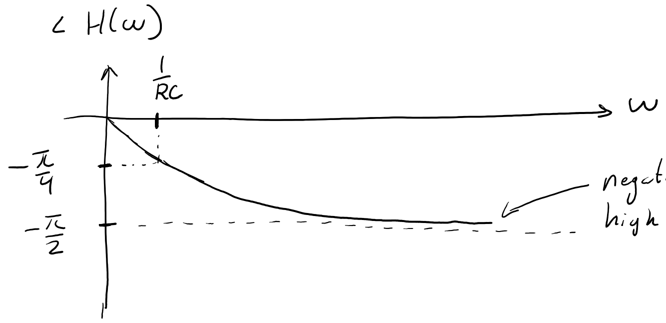


$$H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega CR}$$

$$|H(\omega)| = \left| \frac{1}{1 + j\omega CR} \right| = \frac{1}{\sqrt{1^2 + (\omega CR)^2}}$$

$$\angle H(\omega) = -\tan^{-1}(\omega RC)$$





negative phase shift at high frequencies

High Pass Filter

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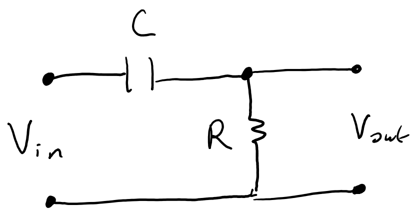
- High Pass Filter

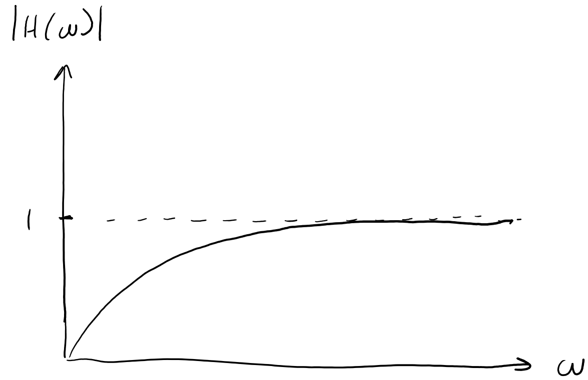
Given:

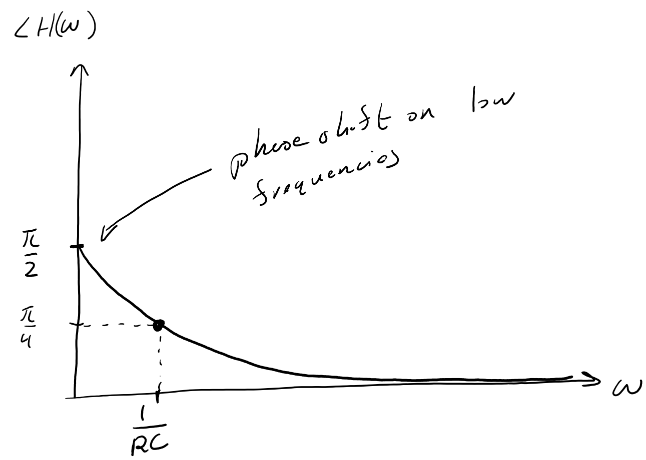


$$H(\omega) = \frac{R}{R + 1/j\omega C} = \frac{j\omega CR}{1 + j\omega CR}$$

$$|H(\omega)| = \frac{\omega CR}{\sqrt{1 + (\omega CR)}}$$

$$\angle H(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega CR)$$





Band Pass Filter

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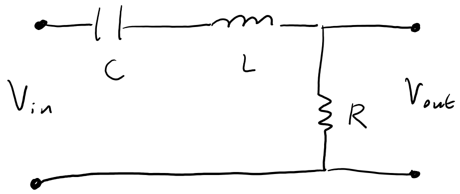
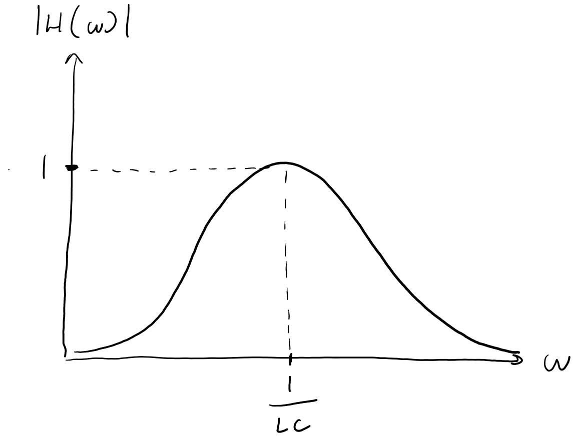
- Band Pass

Given:





$$H(\omega) = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 - \omega^2 LC + j\omega CR}$$

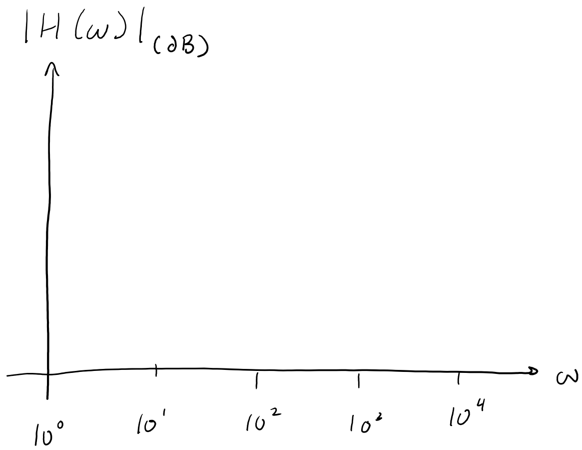
$$|H(\omega)| = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega CR)^2}}$$

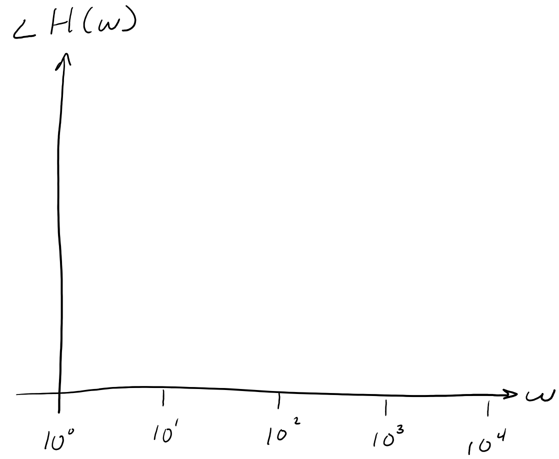
$$\angle H(\omega) =$$

Bode Plots

Tuesday, October 5, 2021 6:33 PM

- Plots of frequency response $H(\omega)$. Since $H(\omega)$ is complex, then there are two plots: Bode plots approximate behavior of these two quantities in a <u>logarithmic</u> scale.





where $|H(\omega)|_{(dB)} = 20\log_{10}|H(\omega)|$

$\angle H(\omega) = \tan^{-1}(\angle H(\omega$

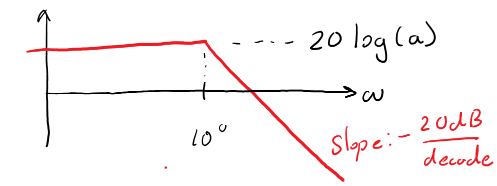
- Identify breakpoint ω_0
- draw behavior before and after ω_0
- note slopes as they appear

- Identify breakpoint ω_0
- Assume end behavior until/after each decade around ω_0 $(\frac{\omega_0}{10}, 10\omega_0)$
- Draw slope between $(\frac{\omega_0}{10}, 10\omega_0)$

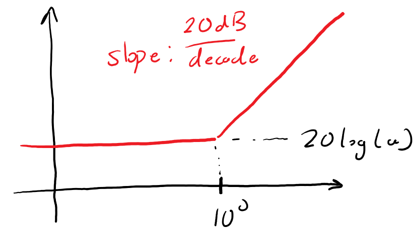
[Magnitude] General Bode Plot Factors

Thursday, October 7, 2021 6:50 PM

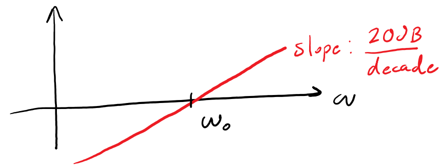
$$H(\omega) = \frac{a}{j\omega}$$



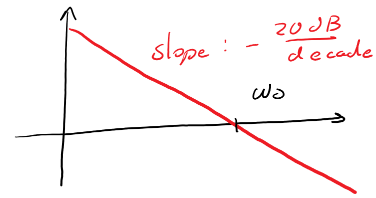
$$H(\omega) = aj\omega$$



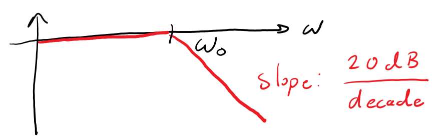
$$H(\omega) = j\frac{\omega}{\omega_0}$$



$$H(\omega) = \frac{1}{j\frac{\omega}{\omega_0}}$$

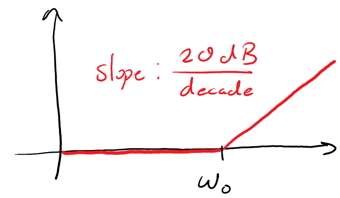


$$H(\omega) = \frac{1}{1+j\frac{\omega}{\omega_0}}$$



$$H(\omega) = 1 + j\frac{\omega}{\omega_0}$$

$$\boxed{|H(\omega)| = \sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}}$$



[Magnitude] General Bode Plot for Pass Filters

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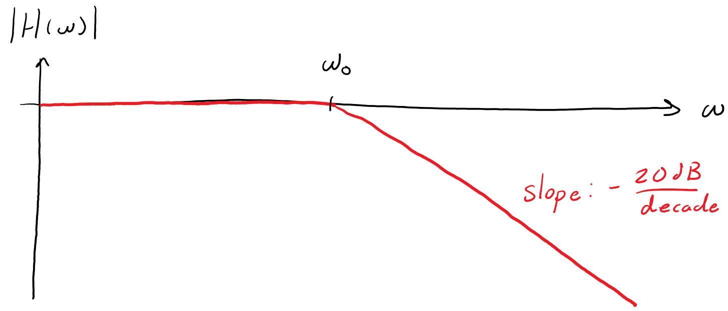
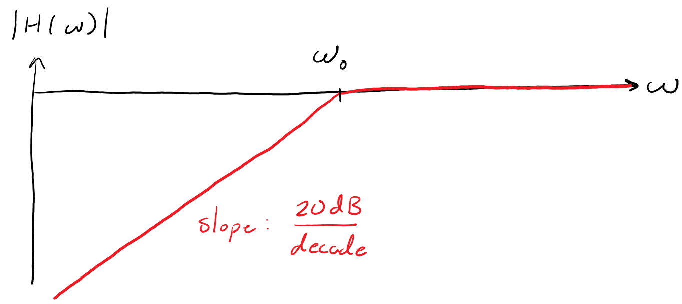
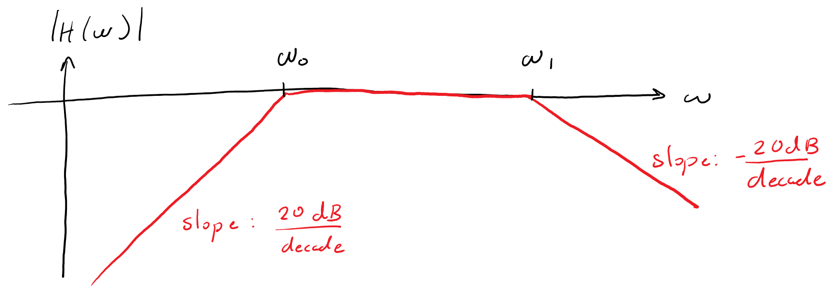
Low Pass: $H(\omega) = \dfrac{1}{\left(1 + j\dfrac{\omega}{\omega_0}\right)}$ $\Big|$ $\omega_0 = \dfrac{1}{RC}$



High Pass: $H(\omega) = \dfrac{j\dfrac{\omega}{\omega_0}}{1 + j\dfrac{\omega}{\omega_0}}$ $\Big|$ $\omega_0 = \dfrac{1}{RC}$



Band Pass: $H(\omega) = \dfrac{j\dfrac{\omega}{\omega_0}}{\left(1 + j\dfrac{\omega}{\omega_0}\right)\left(1 + j\dfrac{\omega}{\omega_1}\right)}$



[Phase] General Bode Plot Factors

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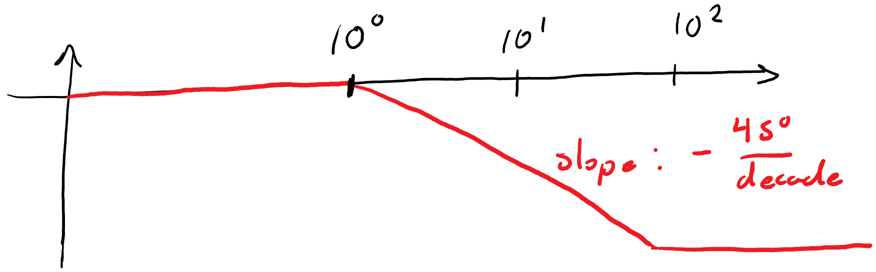
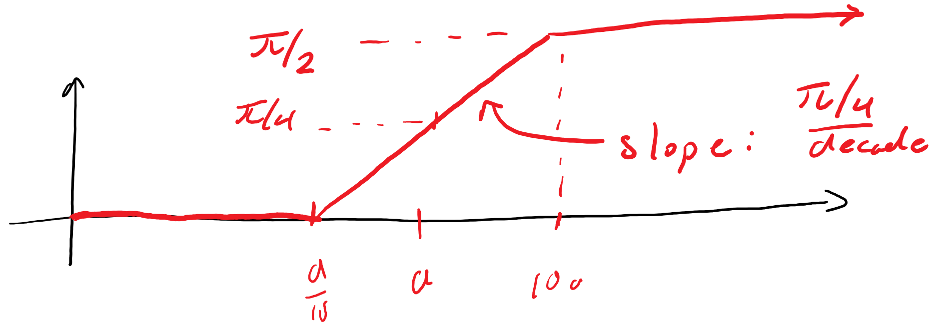
$H(\omega) = \dfrac{a}{j\omega}$, $\tan^{-1}\left(\dfrac{a}{\omega}\right)$



$H(\omega) = \dfrac{\omega}{a}$, $\tan^{-1}\left(\dfrac{\omega}{a}\right)$



Bode Plots for Basic Terms

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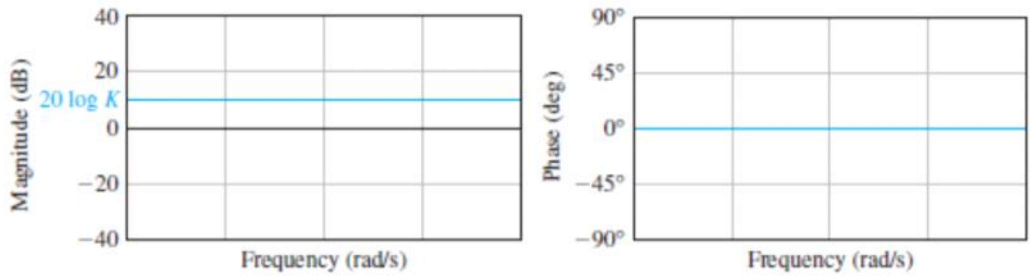
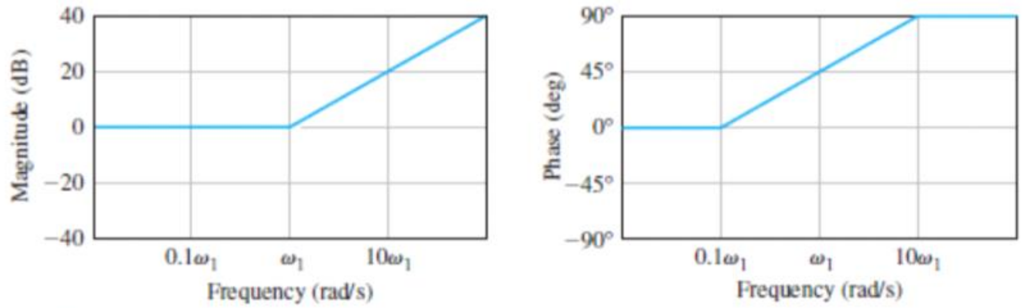
Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude $20 \log_{10} \lvert G(j\omega) \rvert$	Phase $\phi(\omega)$

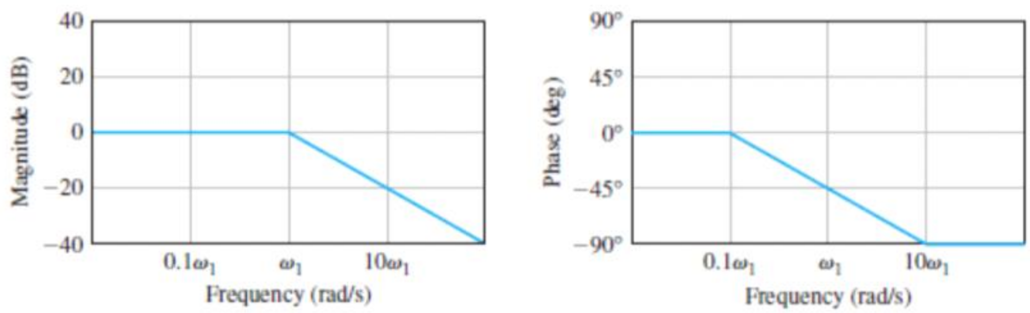
1. Gain, $G(j\omega) = K$



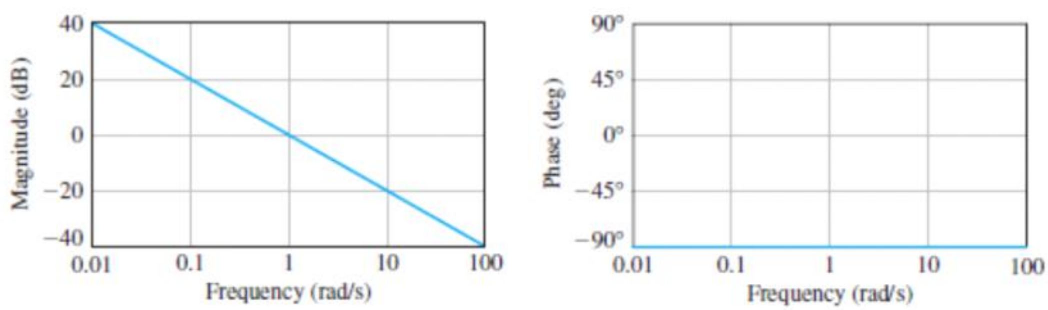
2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$



3. Pole, $G(j\omega) = (1 + j\omega/\omega_1)^{-1}$



4. Pole at the origin, $G(j\omega) = 1/j\omega$



Fourier Series

Tuesday, October 12, 2021 6:26 PM

Recap:

- Given some system:

$$x(t) \longrightarrow \boxed{H(\omega)} \xrightarrow{y(t)} \qquad y(t) = H(\omega)x(t)$$

if $x(t) = A\cos(\omega t + \phi)$, $y(t) = A\,|H(\omega)|\cos(\omega t + \phi + \angle H(\omega))$

Def Fourier Series method applies superposition principle to a combination combination of sinusoids (complex exponentials)

- Suppose the input $f(t)$ as $f(t) = \sum_{k=0}^{\infty} C_k e^{j\omega_k t}$

then the output $g(t) = \sum_{k=0}^{\infty} |H(\omega_k)|\, C_k e^{j[\omega_k t + \angle H(\omega_k)]}$

Idea Any periodic signal can be expressed as an infinite linear combination of sinusoids.

thus: $f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$ where ω_0 is the fundamental frequnecy of the signal

$\omega_0 = \frac{2\pi}{T_0}$ where T_0 is the fundamental period of the signal

and: $C_n = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jn\omega_0 t}\, dt$ for the specific nth coefficient

Specifically: $\lim_{n\to\infty} \int_{T_0} f(x) - \sum_{K-N}^{N} c_k e^{jk\omega_0 t}\, dt = 0$

Special Fourier Series Properties

Thursday, October 14, 2021 7:04 PM

1) Parseval Theorem: "Average power" of a signal is conserved in the spectrum

"average power" = $\frac{1}{T_0} \int_{T_0} |f(t)|^2 \, dt$ and spectrum is the sequence of coefficients

$$= \sum_{k=-\infty}^{\infty} |C_k|^2$$

2) If $f(t) = \sum_k C_k e^{jk\omega_0 t}$, then $f^*(t) = \sum_k C_k^* e^{-jk\omega_0 t}$ which means:

thus, if $f(t)$ is real, then $f(t) = f^*(t)$ and $C_k = C_{-k}^*$

and: $|C_k| = |C_{-k}^*| \rightarrow |C_k| = |C_{-k}|$

3) Shifting property:

if $g(x) = f(x-c)$ then

$g_k = f_k \cdot e^{-jk\omega_0 \cdot c}$ where g_k, f_k are the k^{th} coefficients of g, f

and $g(x) = f(x) + c$ then

$g_0 = f_0 + c$

4) Integral / Derivative Property:

if $f(t)$ has coefficients c_k then $\int f(t)\, dt$ has coefficients $\frac{C_k}{jk\omega_0}$ and C_0

if $f(t)$ has coefficients c_k then $\frac{df}{dt}(t)$ has coefficients $C_k \cdot jk\omega_0$

5) Time reverse property:

if $F(t)$ has coefficients C_k then $F(-t)$ has coefficients F_{-n}

Finding the Coefficients of Fourier Series, Integration

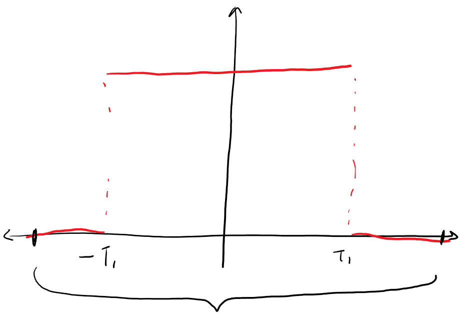
Tuesday, October 19, 2021 6:34 PM

Remember: $f(t) = \sum_{k=-\infty}^{\infty} c_k{}^{jk\omega_0 t}$

$c_n = \frac{1}{T_0} \cdot \int_{T_0} f(t) \cdot e^{-jn\omega_0 t} \, dt$

where $T_0 = \frac{2\pi}{\omega_0}$

Example: given: $f(t)$ = square wave



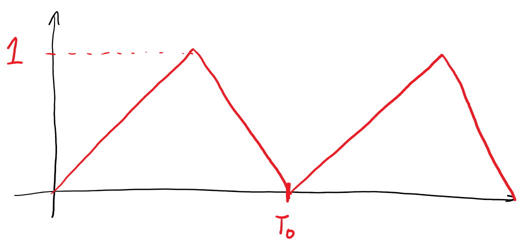
$$c_0 = \frac{1}{T_0} \int f(t)\, dt = \frac{1}{T_0} \cdot 2T_1 = \frac{2T_1}{T_0}$$

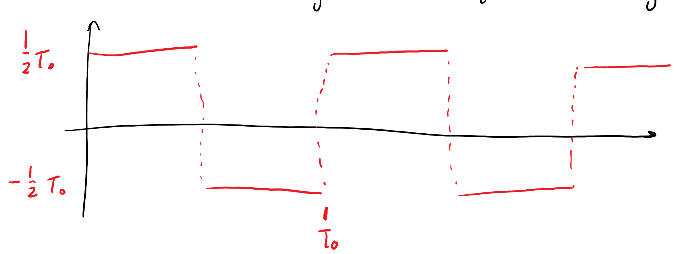
$$c_k = \frac{1}{T_0} \int f(t)\, e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t} dt = -\frac{1}{T_0} \cdot \frac{e^{-jk\omega_0 t}}{jk\omega_0 t} \Bigg]_{-T_1}^{T_1}$$

$$= -\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{2j} \left(\frac{2}{T_0 k\omega_0} \right) = \frac{2}{T_0 k\omega_0} \sin(k\omega_0 T_1)$$

$$f(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{2}{T_0 k\omega_0} \sin(k\omega_0 T_1) \cdot e^{jk\omega_0 t} \right) + \frac{2T_1}{T_0}$$

Example: given $f(t)$ = triangle wave then $f(t)$ is the integral of a square wave $g(t)$:





so $f(t) = \int g(t)\, dt$

if $g(t) = \sum_{\substack{k-\infty \\ k\neq 0}}^{\infty} c_k e^{jk\omega_0 t} + 0$

then $f(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} c_k' e^{jk\omega_0 t} + 0$

$$f(t) = \int \sum_{\substack{k-\infty \\ k\neq 0}}^{\infty} c_k e^{jk\omega_0 t} dt = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \int c_k e^{jk\omega_0 t} dt = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} c_k \frac{e^{jk\omega_0 t}}{jk\omega_0} + C$$

$$f(t) = \int \sum_{\substack{k-\infty \\ k \neq 0}} C_K e^{-jk\omega_0 t} \, dt = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \int C_k e^{-jk\omega_0 t} \, dt = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} C_k \frac{e^{jk\omega_0 t}}{jk\omega_0} + \underbrace{C}_{\text{DC component}}$$

$$\text{so} \quad c'_K = \frac{C_k}{jk\omega_0} \qquad \text{thus} \quad f(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} e^{jk\frac{\pi}{2}} \cdot \frac{2}{\cancel{4} \cdot k \cdot \frac{2\pi}{\cancel{4}}} \cdot \sin\left(k \cdot \frac{2\pi}{4} \cdot 2\right) e^{jk\frac{2\pi}{4} \cdot t} + \frac{1}{2}$$

Finding the Coefficients of Fourier Series, Simpler

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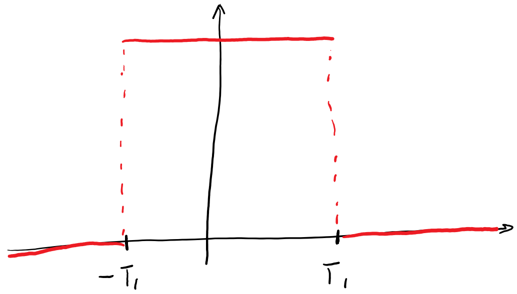
Square wave:



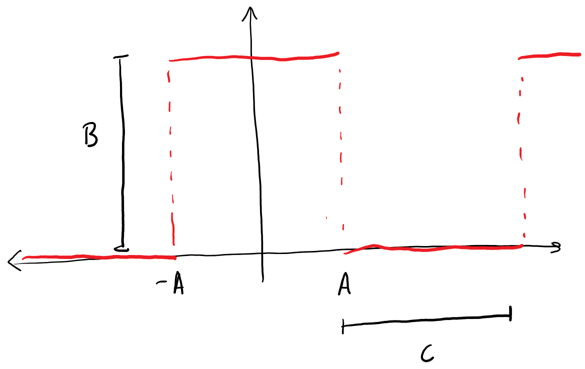
$T_0 \,,\; \omega_0 = \frac{2\pi}{T_0}$

$$c_0 = \frac{2T_1}{T_0} \qquad c_k = \frac{2}{T_0 k \omega_0} \sin(k\omega_0 T_1)$$

$$f(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(\frac{2}{T_0 k \omega_0} \sin(k\omega_0 T_1) e^{jk\omega_0 t} \right) + \frac{2T_1}{T_0}$$

Monday, October 25, 2021 5:40 PM

Given a square wave:



$$T = 2A + C \rightarrow \omega_o = \frac{2\pi}{2A+C}$$

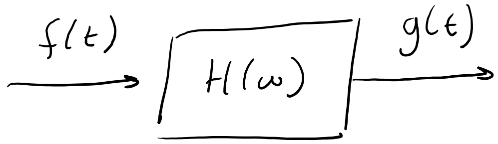
$$F_0 = \frac{1}{T}\int_T f(t)dt = \frac{1}{T}\int_T B\,dt = \frac{2AB}{2A+C}$$

$$F_n = \frac{B}{n\pi}\sin(n\omega_o A) = \frac{B}{n\pi}\sin\left(\frac{2\pi n A}{2A+C}\right)$$

Fourier Series As Input/Output to System

Thursday, October 21, 2021 6:42 PM

Given the system:



if $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

then $g(t) = \sum_{k=-\infty}^{\infty} |H(k\omega_0)| c_k e^{jk\omega_0 t + \angle H(k\omega_0)}$

Fourier Transform

Tuesday, November 2, 2021 6:32 PM

Idea: We can find a sinusoidal representation for any non-periodic signal

Approach: 1) Start with the FS of periodic signal
2) take limit to get more general representation in terms of Fourier Transform

Idea: when $\omega_0 \to 0$ for the new sinc function for coefficients:
then the function $f(t)$ repeats further and further into $\pm\infty$. thus it is no longer periodic

- As $\omega_0 \to 0$, the signal $f(t)$ loses its periodic character and looks like a non-periodic shape.

Note: As $\omega_0 \to 0$, the sampled values for coefficients become closely packed and resemble a continous function. We call this function the Fourier Transform

Def: Formally we can derive the Fourier Transform by:

$$FT(f(t)) = F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Def: If $\hat{f}(t)$ is a periodic signal, and $f(t)$ is the corresponding non-periodic signal:

$$\hat{f}(t) = \sum C_K e^{jk\omega_0 t} = \frac{1}{2\pi} \sum \omega_0 F(jk\omega_0)\, e^{jk\omega_0 t}$$

when $\omega_0 \to 0$, then $\hat{f}(t) \to f(t)$

thus $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

Summary: For non-periodic signal $f(t)$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega \qquad F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Special Fourier Transform Properties

Thursday, November 4, 2021 7:01 PM

1) Convergence : $\lim_{T\to\infty} \int_{-\infty}^{\infty} \left| x(t) - \frac{1}{2\pi}\int_{-T}^{T} X(j\omega)\, e^{j\omega t}\, d\omega \right|^2 = 0$ provided that $\int_{-\infty}^{\infty} p(t)\,dt < \infty$

2) Time Shift Property : Given $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$

then $x(t - t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, e^{-j\omega t_0}\, d\omega$

thus $x(t - t_0)$ has FT function $X(j\omega)\cdot e^{-j\omega t_0}$

a time shift in the time domain corresponds to a phase shift in frequency domain by ωt_0

3) Multiplication by Complex: Given $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

then $x(t)\cdot e^{j\omega_0 t}$ has FT $= \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$

thus $x(t)\cdot e^{j\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega-\omega_0))\, e^{j\omega t}\, d\omega$ and has FT $= X(j(\omega-\omega_0))$

a phase shift in the time domain results in a spectrum shift in the frequency domain by ω_0

4) Time - Frequency Duality: if $FT(x(t)) = X(j\omega)$ then

$\int x(t)\, dt = X(0)$

and if $FT(x(t)) = X(j\omega)$ and $X(j\omega) = y(t)$ then

$FT(y(t)) = Y(j\omega) = x(t)$

5) Parseval's Theorem $\int x(t)^2\, dt = \frac{1}{2\pi}\int |X(j\omega)|^2\, d\omega$ for real signals

Energy is conserved in the Transform

6) Derivative & Integration: given $x(t)$ and $FT(x(t)) = X(j\omega)$

then $FT\left(\frac{\partial}{\partial t} x(t)\right) = j\omega X(j\omega)$

and $FT\left(\int x(t)\,dt\right) = \frac{1}{j\omega} X(j\omega) + \pi X(0)\, \delta(\omega)$

where the area of $\delta(\omega) = 1$

$$FT^{-1}\left(FT\left(\int x(t)\,dt\right)\right) = \frac{1}{2\pi}\int \frac{1}{j\omega} X(j\omega)\, e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int \pi X(0)\, \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int \frac{1}{j\omega} X(j\omega)\, e^{j\omega t}\, d\omega + \frac{1}{2\pi}\left(\pi X(0)\right)$$

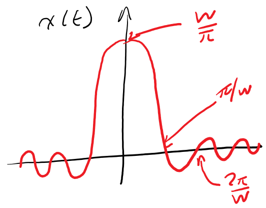
$$= FT^{-1}\left(\frac{1}{j\omega} X(j\omega)\right) + \frac{1}{2} X(0)$$

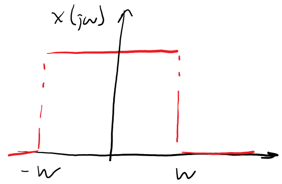
7) Time Scaling: if $FT(x(t)) = X(j\omega)$ then $FT(x(at)) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

Ideal Pulse, Delta

Tuesday, November 9, 2021 7:21 PM

With $x(t)$ and $X(j\omega)$:

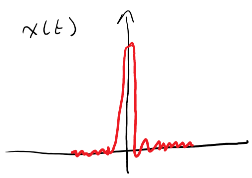


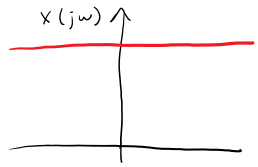


as $\lim_{W\to\infty}$, $X(j\omega)$ becomes all frequencies,

$x(t)$ becomes a narrow pulse.

Thus as $W \to \infty$



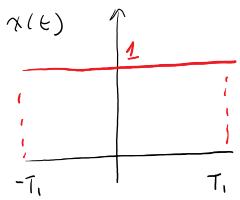


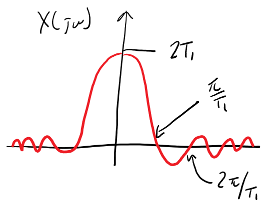
$x(t)$ is called an ideal pulse which is infinite at the origin and 0 everywhere else.

symbol: $\delta(t)$

and thus: $FT(\delta) = 1$ and $FT^{-1}(1) = \delta$

With $x(t)$ and $X(j\omega)$:





as $\lim_{T_1\to\infty}$, $x(t)$ becomes a constant frequency

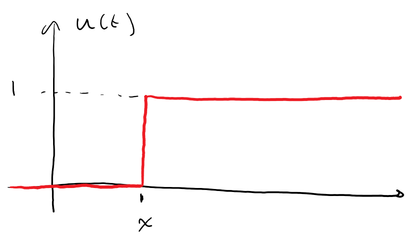
$X(j\omega)$ becomes infinite at $\omega = 0$ and 0 everywhere else

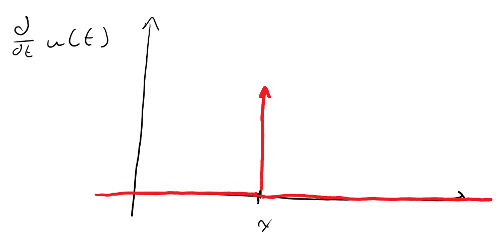
so $X(j\omega) = \delta(j\omega)$

and thus: $FT(1) = 2\pi\,\delta(\omega)$, $FT^{-1}(2\pi\,\delta) = 1$

<u>Note</u>: $\delta(j\omega)$ has area of 1!

<u>Idea</u>: We can also define $\delta(j\omega)$ as derivative of step function!





Fundamental Properties of Delta-Function (Integral)

Thursday, November 18, 2021 7:16 PM

Def Fundamental Property of δ-functions:

$$\int f(t) \cdot \delta(t) dt = f(0)$$

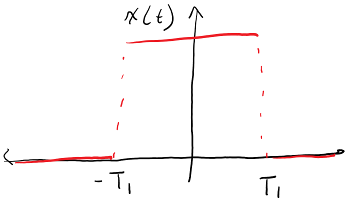
and $$\int f(t) \cdot \delta(t - t_0) dt = f(t_0)$$

Def $\delta(at) = \frac{1}{|a|} \delta t$, $\delta(-t) = \delta(t)$

Example with Square Wave

Tuesday, November 9, 2021 6:39 PM

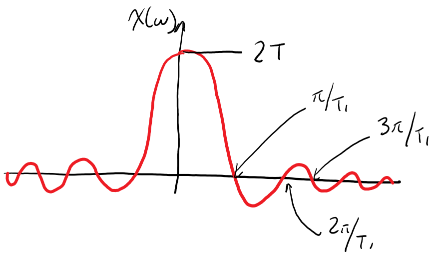
Given the signal $x(t)$:



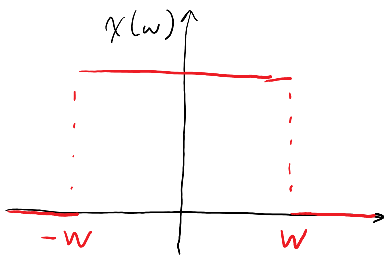
$$FT(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T_1}^{T_1}$$

$$= \frac{e^{-j\omega T_1} - e^{j\omega T_1}}{-j\omega} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$

$$= 2T_1 \frac{\sin \omega T_1}{\omega T_1}$$



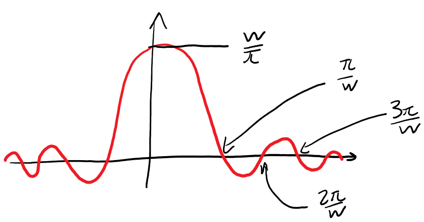
Given the fourier transform $X(j\omega)$:



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, dt = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega t}}{jt} \right]_{-W}^{W} = \frac{1}{2\pi} \left[\frac{e^{jWt} - e^{-jWt}}{jt} \right]$$

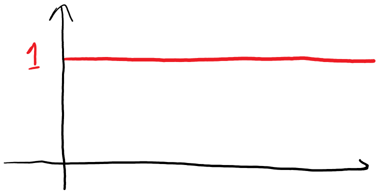
$$= \frac{1}{2\pi} \cdot W \cdot 2 \cdot \left[\frac{e^{jWt} - e^{-jWt}}{2jWt} \right] = \frac{W}{\pi} \frac{\sin(Wt)}{Wt}$$



Example with Step Function

Tuesday, November 16, 2021 7:28 PM

Ex Given $u(t)$:



and $f(t) = u(t) \cdot e^{-j\frac{t}{T}}$

$$FT(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-j\frac{t}{T}} \cdot e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\left(\frac{1}{T} + j\omega\right)t}\, dt = \left. \frac{-e^{-\left(\frac{1}{T} + j\omega\right)}}{\frac{1}{T} + j\omega} \right|_{0}^{\infty}$$

$$= \frac{1}{\frac{1}{T} + j\omega} = \frac{1/T}{1/T^2 + \omega^2} - j\frac{\omega}{1/T^2 + \omega^2}$$

Ex Given $u(t)$ from above find $FT(u(t))$

$u(t) = \lim_{T\to\infty} f(t)$ thus $FT(u(t)) = \lim_{T\to\infty} \frac{1/T}{1/T^2 + \omega^2} - j\frac{\omega}{1/T^2 + \omega^2}$

$$= \begin{cases} -\frac{j}{\omega} & \text{for } \omega \neq 0 \\ \infty & \text{for } \omega = 0 \end{cases}$$

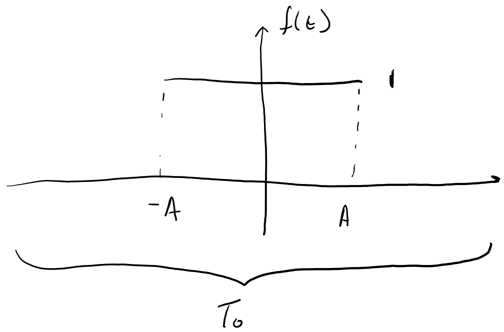
$$= \begin{cases} -\frac{j}{\omega} & \text{for } \omega \neq 0 \\ \pi\delta(\omega) & \text{for } \omega = 0 \end{cases}$$

$$= -\frac{j}{\omega} + \pi\delta(\omega)$$

Example with Square Wave

Tuesday, November 2, 2021 7:08 PM

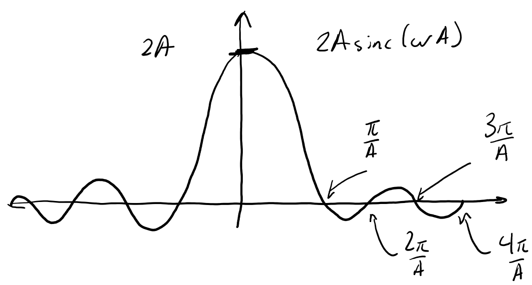
Ex:



$$f(t) = \sum C_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$C_k = \frac{\sin(k\omega_0 A)}{k\pi} \qquad C_0 = \frac{2A}{T_0}$$

$$C_k = \frac{\omega_0 \sin(k\omega_0 A)}{k\pi\omega_0}$$



$$T_0 C_k = \frac{2\sin(k\omega_0 A)}{k\omega_0}$$

$$T_0 C_k = \frac{2A\sin(\omega A)}{A\omega}\Big|_{\omega = k\omega_0}$$

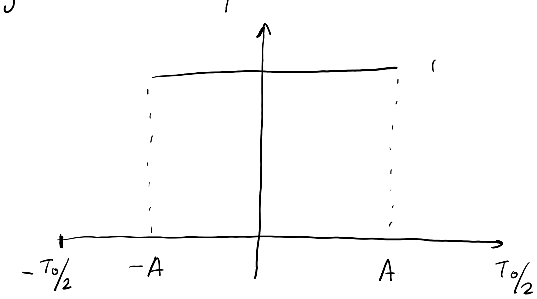
$$= 2A\,\text{sinc}(\omega A)\,\big|_{\omega = k\omega_0}$$

$$C_k = \frac{1}{T_0} 2A\,\text{sinc}(\omega A)$$

thus the coefficients sample values from the sinc function. the location of the samples are dependent on ω_0.

FT: when $\omega_0 \to 0$, then the samples of values approach $2A\,\text{sinc}(\omega A)$ thus the Fourier Transform $= 2A\,\text{sinc}(\omega A)$

Formally: given the periodic $\hat{f}(t)$ and non-periodic $f(t)$:



Normally: $C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{T_0}\int_{-\infty}^{\infty} f(t)\, e^{-jk\omega_0 t}\, dt$$

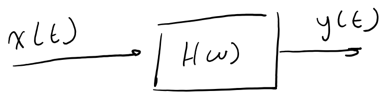
$$= \frac{1}{T} F(jk\omega_0)$$

$$\hat{f}(t) = \sum C_k e^{jk\omega_0 t} = \sum \omega_0 F(jk\omega_0)\, e^{jk\omega_0 t}$$

Fourier Transform as Input/Output to System

Thursday, November 4, 2021 6:41 PM

Given the system:



if $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$ then $y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) H(\omega)\, e^{j\omega t}\, d\omega$

$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, d\omega$

$Y(j\omega) = X(j\omega) \cdot H(\omega)$

General Method of Solving FT Systems

Thursday, November 4, 2021 6:51 PM

Given a signal $x(t)$, and system $H(\omega)$

1) compute $FT(x(t)) \rightarrow X(j\omega)$

2) Multiply $X(j\omega)H(\omega) = Y(j\omega)$

3) compute $FT^{-1}(Y(j\omega)) \rightarrow y(t)$

Summary of Properties of FT

Tuesday, November 23, 2021 6:35 PM

Time Scaling: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

Time Shift: $x(t - t_0) \longleftrightarrow X(j\omega)\, e^{-j\omega t_0}$

Mul by Complex: $x(t)\, e^{j\omega_0 t} \longleftrightarrow X(j(\omega - \omega_0))$

Time Frequency Duality: if $x(t) \longleftrightarrow X(j\omega)$ then

$\int x(t)\,dt = X(0)$

Derivative / Integral: if $x(t) \longleftrightarrow X(j\omega)$

Convolution

Tuesday, November 23, 2021 6:41 PM

<u>Def</u> Given functions $f(t) \longleftrightarrow F(j\omega)$, $g(t) \longleftrightarrow G(j\omega)$ then:

$$f \circ g(t) = \int_{-\infty}^{\infty} f(x)\, g(t-x)\,dx = \int_{-\infty}^{\infty} g(x)\, f(t-x)\,dx$$

and:

$$FT(f \circ g) = F(j\omega) \cdot G(j\omega)$$

$$FT(f \cdot g) = \frac{1}{2\pi} F \circ G$$

<u>Idea</u> Given an input $x(t)$ and Linear system $h(t)$:

$$y(t) = x(t) \circ h(t) = \int_{-\infty}^{\infty} x(v)\, h(t-v)\,dv = \int_{-\infty}^{\infty} h(v)\, x(t-v)\,dv$$

Convolution Property of Delta Function, Examples

Tuesday, November 23, 2021 6:43 PM

Def Given some $f(t)$:

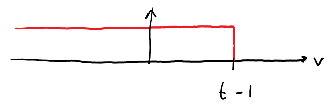
$$f \circ \delta(t) = f(t)$$

Ex Given $x(t) = u(t-1)$ and $h(t) = e^{-t}u(t)$

then $x \circ h = \int_{-\infty}^{\infty} h(v)\, x(t-v)\, dv$

1) Flip Signal : $x(-v)$:

2) Shift Signal : $x(t-v)$:



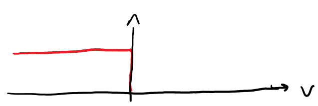
Thus: $x \circ h = \begin{cases} 0 & t<1 \\ \int_0^{t-1} h(v)\, dv & t>1 \end{cases} = \begin{cases} 0 & t<1 \\ -e^{-v}\Big|_0^{t-1} & t>1 \end{cases} = \begin{cases} 0 & t<1 \\ 1-e^{-t+1} & t>1 \end{cases}$

Then: $x \circ h = (1-e^{-t+1})\,u(t-1)$

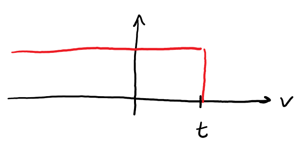
Ex Given $x(t) = u(t)$ and $h(t) = u(t)$

$x \circ h = \int_{-\infty}^{\infty} x(v)\, h(t-v)\, dv$

1) Flip : $h(-v)$:



2) Shift : $h(t-v)$:

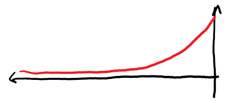


thus : $x \circ h = \begin{cases} 0 & t<0 \\ t & t>0 \end{cases} = t \cdot u(t)$

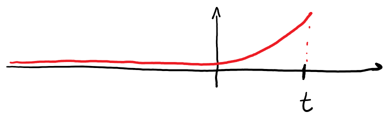
Ex Given $x(t) = e^{-at}u(t)$ $h(t) = e^{-bt}u(t)$

$x \circ h = \int_{-\infty}^{\infty} x(v) \cdot h(t-v)\, dv$

1) Flip :



2) Shift :



thus: $x \circ h = 0$ when $t < 0$,

$$\int_0^t e^{-av} \cdot e^{-b(t-v)} \, \partial v$$

$$= \int_0^t e^{-av} \cdot e^{-bt+bv} \, \partial v$$

$$= e^{-bt} \cdot \int_0^t e^{(b-a)v} \, \partial v$$

$$= e^{-bt} \cdot \frac{1}{b-a} e^{(b-a)v} \Big|_0^{v=t}$$

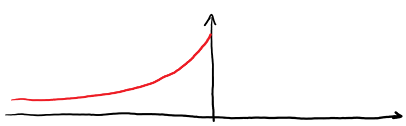
$$= e^{-bt} \cdot \left(\frac{1}{b-a} e^{(b-a)t} - \frac{1}{b-a} \right)$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}$$

Ex Given $x(t) = u(t-1) - u(t-2)$ $\quad h(t) = e^{-t} u(t)$

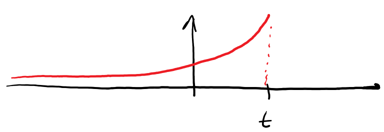
$$x \circ h = \int x(v) h(t-v) \, \partial v$$

1) Flip :



$x(t)$:

2) Shift :



thus: $x \circ h = 0$ when $t < 1$,

$$\int_1^t e^{-(t-v)} \, \partial v \quad \text{when } 1 < t < 2 = \left[e^{-t+v} \right]_1^t = 1 - e^{-t+1}$$

$$\int_1^2 e^{-(t-v)} \, \partial v \quad \text{when } t > 2 = \left[e^{-t+v} \right]_1^2 = e^{-t+2} - e^{-t+1}$$

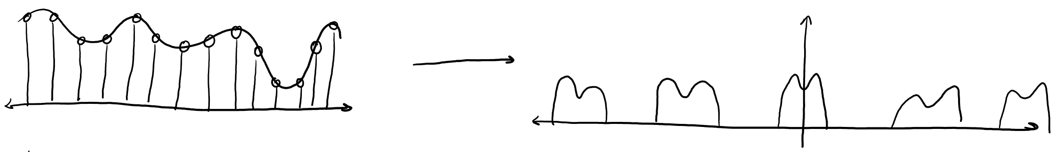
Sampling Theorem

Tuesday, November 23, 2021 7:09 PM

Def The Sampling Theorem:

We can take a time dependent signal into a set of discrete values by sampling the signal at small time intervals.

Specifically: because an audio signal has finite range (bandwidth) then extracting samples of the signal in time copies the spectrum in frequency



Thus: sampling a signal in time domain corresponds to replicating its spectrum in frequency domain across the frequencies.

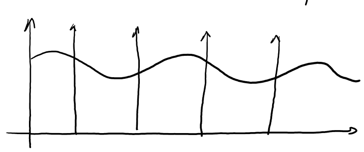
Proof: Given some periodic signal $x(t) = \sum C_k e^{j\omega_0 k t}$

then: $FT(x(t)) = \sum C_k \cdot 2\pi\,\delta(\omega - k\omega_0)$

if $y(t) = \sum \delta(\omega - k\omega_0)$ then $y(t) = \sum C_k e^{jk\omega_0 t}$ where $C_k = \frac{1}{T}\int \delta(t)\, e^{-jk\omega_0 t}\,dt = \frac{1}{T}$

and $FT(y(t)) = \frac{1}{T}\sum 2\pi\,\delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum \delta(\omega - k\omega_0)$

thus: if we sample a periodic signal with ideal pulses:



$\Big\}\ z(t) = x(t)\cdot \sum \delta(t - kT)$

then: $FT(z(t)) = \frac{\omega_0}{2\pi} X(\omega) \circ \sum \delta(\omega - k\omega_0)$

$$= \frac{\omega_0}{2\pi} \sum X(\omega - k\omega_0)$$

which is the spectrum repeated every $k\omega_0$

Def Then: the sampling frequency $\omega_0 = \frac{2\pi}{T}$, $\omega_0 > 2W$. where w is the highest frequencies of the input signal.

Alternatively: Given some signal $x(t)$ with maximum bandwidth $W < \infty$

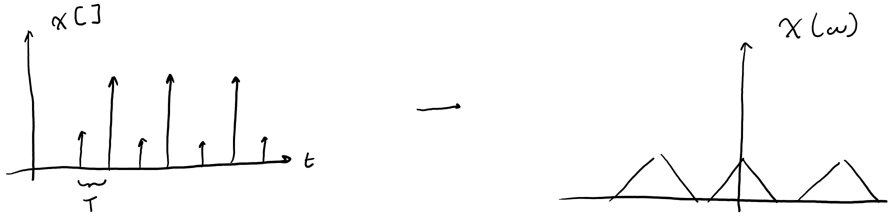
then $\omega_0 > 2W$ is the sampling frequency.

How to Apply The Sampling Theorem

Sunday, December 5, 2021 6:25 PM

<u>Def</u> Given samples $x[n]$ reconstruct $x(t)$:

1) construct the impulse train with each $x[n]$ weighted to a δ funtion



2) Apply a low pass filter.

3) Take inverse fourier to reconstruct the original signal.