What are Diff Eqs

Friday, September 24, 2021 12:20 PM

Algebraic

$$ax^2 + bx + c = 0$$

we want to solve for x which is a number

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Involves unknown variables.

Differential

$$f'(x) - x = 0$$

we want to solve for $f(x)$, which is a function.

$$f'(x) = x \rightarrow f(x) = \int x \, dx$$

$$f(x) = \frac{x^2}{2} + C$$

special soln general soln

Involves unknown functions and derivatives

Diff Eq Formats

Monday, September 27, 2021 12:06 PM

$f'(\underline{x}) - \underline{x} = 0$

independent variable

$y' - x = 0$

dependent variable

$f'(t) - t = 0$

all representations are equivalent

Classification of Diff Eqs

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1) Linear / Non-linear

Linear iff there are no powers on the dependent variable and there cannot be products of derivatives of the dependent variable

Eg: $2y + 3y' - xy'' = 0$ (Linear)

$2y^2 + 3y'y'' - x(y'')^3 = 0$ (Non-Linear)

2) Homogeneous / Non-homogenous

Homogenous iff 0 is on the right hand side. (no function of x)
if it is possible to make a non-0 term trough algebra, non-homogenous

3) Order

Highest derivative level in the equation

$y' \to$ 1st Order
$y'' \to$ 2nd Order
etc.

General Model of 2nd Order Linear Diff Eq

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$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

where $p(x), q(x), r(x), s(x)$ are known functions

Solutions to Diff Eqs

Monday, September 27, 2021 12:26 PM

Solutions to Diff Eqs are functions $y = y(x)$

Verify a Solution

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- Take the derivatives of the solution
- Substitute the derivatives
- Solve, if the solution is consistent, it is correct.

Implicit Derivative

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given $F(x, y) = \ldots\ldots$

$$\frac{\partial y}{\partial x} = - \frac{\partial F/\partial x}{\partial F/\partial y}$$

alternatively:

$$\frac{\partial y}{\partial x}(F(x)) = \frac{\partial F}{\partial x}, \quad \frac{\partial y}{\partial x}(F(y)) = \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x}$$

ie: $\frac{\partial y}{\partial x}(2y^3 + 8x^2 - 10) = 6y^2 \frac{\partial y}{\partial x} + 16x$

Existence and Uniqueness

Wednesday, September 29, 2021 12:04 PM

Existence: there exists a solution to the Diff Eq:

Ex: $y' = -y$

$y = e^{-x}$, $y' = -e^{-x}$

there exists a solution to the Diff Eq

Uniqueness: the solution is the only solution

Ex: $y' = -y$

$y = e^{-x}$, $y = C \cdot e^{-x}$, $y = e^{-x+C}$

the solution $y = e^{-x}$ is not unique

Note: adding additional conditions can narrow a single unique solution

Initial Value Problem

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If we have a 1st order Diff Eq:

$$\begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

Or a nth order Diff Eq:

$$\begin{cases} y^n = F(x, y', y'', \dots y^{n-1}) \\ n \text{ additional starting conditions} \end{cases}$$

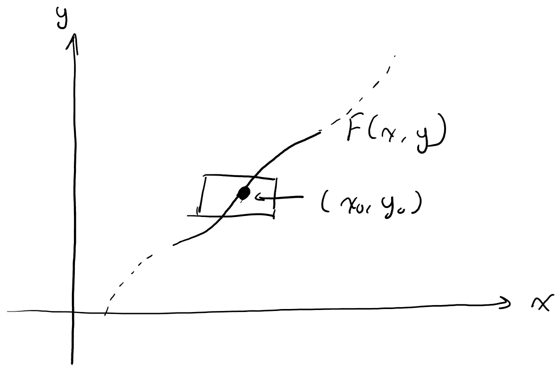
Theorem (Existence & Uniqueness of 1st Order Diff Eq)

Wednesday, September 29, 2021 12:41 PM

Given the 1st order Diff Eq:

$$\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

where $F(x,y)$ and $\frac{\partial F}{\partial y}(x,y)$ are continous in a rectangle around (x_0, y_0)



Then: there exists a unique solution in the rectangle.

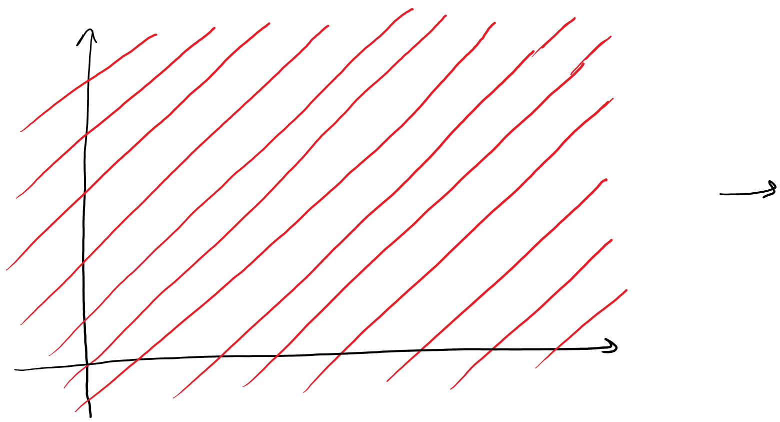
Direction Fields, Isoclines

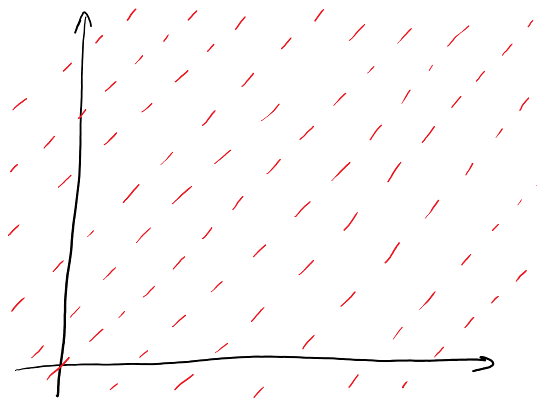
Friday, October 1, 2021 12:27 PM

- Given some Diff Eq, we can draw a family of solutions:

Ex: $y' = 1$, $y = x + C$

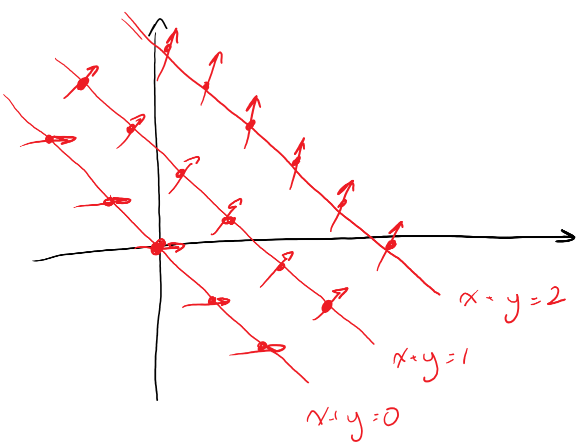
Direction Field:





- For non-homogenous equations: Use Isoclines (level curves)

$y' = x + y$



Solving 1st Order Diff Eqs

Monday, October 4, 2021 12:01 PM

Def 1st order means only y or y'

Separable Equations

Monday, October 4, 2021 12:26 PM

Simple DE : $y' = 1$ $\frac{dy}{dx} = 1$ → simplest separable equation

Def Separable Equations are equations which are:

1) First order equations
2) $\frac{dy}{dx} = g(x) \cdot h(y)$

Solve:

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

$$dy = g(x) \cdot h(y)\, dx$$

$$\frac{dy}{h(y)} = g(x)\, dx$$

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$$

Ex: $\frac{dy}{dx} = \frac{x-5}{y^2}$

$$y^2 dy = (x-5)dx$$

$$\int y^2 dy = \int x - 5\, dx$$

$$\frac{1}{3}y^3 + C = \frac{1}{2}x^2 - 5x + C$$

$$y = \sqrt[3]{\frac{3}{2}x^2 - 15x + C} \quad | \; y \neq 0$$

Linear Equations

Wednesday, October 6, 2021 12:33 PM

<u>Def</u> 1st order Linear Equation are:

1) Linear, 1st order

2) $a(x)y' + b(x)y = c(x)$ <u>or</u> $y' p(x)y = q(x)$

<u>Solve</u>: $y' + \frac{b(x)}{a(x)} y = \frac{c(x)}{a(x)}$ $a(x) \neq 0$

$y' + p(x)y = q(x)$ | $p(x) = \frac{b(x)}{a(x)}$, $q(x) = \frac{c(x)}{a(x)}$ $a(x) \neq 0$

$f(x)\left(y' + p(x)y\right) = f(x)q(x)$ where $f(x) = e^{\int p(x)\,dx}$, $f'(x) = p(x)f(x)$

integration factor, no $+C$

$\frac{d}{dx}\left[f(x)y\right] = f(x)q(x)$

$f(x)y = \int f(x)q(x)\,dx$

<u>Ex</u>: $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x} = x\cos x$

$y' - \frac{2y}{x} = x^2\cos x$ $p(x) = -\frac{2}{x}$ $q(x) = x^2\cos x$ $f(x) = e^{\int -\frac{2}{x}} = e^{-2\ln|x|} = \frac{1}{x^2}$

$\frac{1}{x^2}\left(y' - \frac{2y}{x}\right) = \frac{1}{x^2}x^2\cos x$

$\frac{d}{dx}\left[\frac{1}{x^2}y\right] = \cos x \rightarrow \int \frac{d}{dx}\left[\frac{1}{x^2}y\right]dx = \int \cos x\,dx$

$\frac{1}{x^2}y = \sin x + C \rightarrow y = x^2\sin x + Cx^2$

Initial Value Problem for 1st Order Linear Equations

Friday, October 8, 2021 2:21 PM

- Initial value problem for 1st order Linear equations
  - domain restrictions on coefficients $a(x), b(x), c(x)$
  - existence & uniqueness of solution :

$$y' + p(x)y = q(x) \qquad y(x_0) = y_0$$

$$y' = q(x) - p(x)y$$

$$F(x,y) = q(x) - p(x)y$$ } always continous in y, not always in x

$$\frac{\partial F}{\partial y}(x,y) = -p(x)$$

thus: if $p(x), q(x)$ are continous, then there is a unique solution in some area containing (x_0, y_0)

Exact Equations

Monday, October 11, 2021 3:33 PM

<u>Def</u> Exact equations :

1) Linear, 1st order
2) $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ or $M(x,y)\,\partial x + N(x,y)\,dy = 0$

<u>Thm</u> : To test for exactness

if $M = \frac{\partial F}{\partial x}$, $N = \frac{\partial F}{\partial y}$, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

then if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$, the equation is exact, and $F(x,y)$ exists

<u>Solve</u> : If there exists a function $F(x,y)$ such that:

$\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$ then $F(x,y) = C$ is the solution

if the equation is exact:

1) $\int M\,dx = F(x) + C(y)$
2) $\int N\,dy = F(y) + C(x)$
3) $F(x,y) = \int M\,dx + \int N\,dy + C$
   - add repeated terms only once

Special Integration Factors

Wednesday, October 13, 2021 12:31 PM

- Given the equation:

$$\frac{dy}{dx} = -\frac{M}{N} \quad , \text{ if } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad (\text{not exact})$$

<u>Idea</u> then we can find μ:

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \quad \text{where} \quad M_y = \frac{\partial M}{\partial y}, \; N_x = \frac{\partial N}{\partial x} \quad \text{iff} \quad \frac{M_y - N_x}{N} \text{ is a function of only } x$$

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} \quad \text{where} \quad N_x = \frac{\partial N}{\partial x}, \; M_y = \frac{\partial M}{\partial y} \quad \text{iff} \quad \frac{N_x - M_y}{M} \text{ is a function of only } y$$

then: $\mu M dx + \mu N dy = 0$ is exact

<u>Idea</u> if $\frac{M_y - N_x}{N}$ or $\frac{N_x - M_y}{M}$ are not satisfied then:

Solving 2nd Order LINEAR Diff Eqs

Friday, October 15, 2021 3:54 PM

<u>Def</u> 2nd order linear equations:

$$a(x)y'' + b(x)y' + c(x)y = r(x)$$

$$y'' + p(x)y' + q(x)y = r(x)$$

Constant Coefficients, Homogenous

Friday, October 15, 2021 4:04 PM

- Given the equation: $y'' + ay' + by = 0$

Solution: Don't use integration! Use linear algebra!

if $y = y_1(x)$, $y = y_2(x)$ are solutions, then $y = y_1(x) + y_2(x)$ is also a solution

then y is any linear combination of $y_1(x)$, $y_2(x)$ are solutions iff $y_1(x)$ and $y_2(x)$ are linearly independent ($y_1 \neq \lambda y_2$)

$y = 0$ is the trivial solution (not helpful)

$y = e^{rx}$ is a possible solution, where r must be solved for

- specifically $r^2 + ar + b = 0$ } characteristic equation
  - if $a^2 - 4b > 0$, then $r = r_1, r_2$ and $y_1 = e^{r_1 x}$ $y_2 = e^{r_2 x}$

    then soln: $y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ iff

    y_1, y_2 are LI via $\frac{y_1}{y_2} \neq C$ or $y_1' y_2 - y_1 y_2' \neq 0$ for all x
  - if $a^2 - 4b = 0$, then $r = r_1 = r_2$ and $y_1 = e^{r_1 x}$ $y_2 = x e^{r_2 x}$

    then soln: $y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 x e^{r_2 x}$
  - if $a^2 - 4b < 0$, the r has no real roots and $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

    then soln: $y = C_1 y_1 + C_2 y_2 = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$

    note: α controls magnitude of soln
    β controls frequency of soln

Summary: for the equation: $ay'' + by' + cy = 0$, $ar^2 + br + c = 0$

if $r = r_1, r_2$ then $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$

if $r = r_1 = r_2$ then $y_1 = e^{rx}$, $y_2 = xe^{rx}$

if $r = \alpha + \beta i$ then $y_1 = e^{\alpha x} \cos \beta x$, $y_2 = e^{\alpha x} \sin \beta x$

Constant Coefficients, Specific Non-Homogenous

Monday, October 25, 2021 6:37 PM

Given the equation : $ay'' + by' + cy = r(x)$

1) if $r(x)$ has degree n, then y must have the same degree

solve: plug in y with unknown coefficients and solve for coefficients

2) if $r(x) = e^{kx}$ then:

if k is not a root of $ak^2 + bk + c = 0$ then $y = Ce^{kx}$

if k is a single root of $ak^2 + bk + c = 0$ then $y = Cxe^{kx}$

if k is a double root of $ak^2 + bk + c = 0$ then $y = Cx^2 e^{kx}$

if $r(x) = e^{\alpha x} \sin\beta x$ or $e^{\alpha x} \cos\beta x$ then $\alpha \pm \beta i$

if $ak^2 + bk + c$ has roots $= \alpha + \beta i$ then $y = Ce^{\alpha x} \sin\beta x + De^{\alpha x} \cos\beta x$

if $ak^2 + bk + c$ has roots $\neq \alpha + \beta i$ then $y = Cxe^{\alpha x} \sin\beta x + Dxe^{\alpha x} \cos\beta x$

solve: plug in y with unknown coefficient and solve for coefficients

3) if $r(x)$ = polynomial $\times$ exponential then y = soln for poly $\times$ soln for exp

if y_p is a solution for $ay'' + by' + cy = r(x)$ and y_1, y_2 is a solution for $ay'' + by' + cy = 0$

then the solutions for $ay'' + by' + cy = r(x)$ is $y = y_p + C_1 y_1 + C_2 y_2$

Superposition Principle

Friday, October 29, 2021 2:19 PM

<u>Thm</u> Given a second order linear equation:

if y_1 solves $y'' + ay' + by = r_1(x)$

y_2 solves $y'' + ay' + by = r_2(x)$

then $y_1 + y_2$ solves $y'' + ay' + by = r_1(x) + r_2(x)$

Wronskian

Wednesday, November 3, 2021 8:49 PM

The wronskian of two solutions y_1, y_2 : $W(y_1, y_2) = y_1 y_2' - y_1' y_2$

Constant Coefficients, General Non-Homogenous

Friday, October 29, 2021 2:22 PM

· Given the equation $y'' + ay' + by = r(x)$ for some general $r(x)$:

Idea Variation of parameters: Let $y = v_1(x)\, y_1 + v_2(x)\, y_2$ where y_1, y_2 are soln to homogenous equation

Goal solve for $v_1(x)$, $v_2(x)$:

$$y = v_1 y_1 + v_2 y_2 \qquad \text{let} \quad v_1' y_1 + v_2' y_2 = 0$$

$$y' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

$$= v_1 y_1' + v_2 y_2'$$

$$y'' = v_1' y' + v_1 y'' + v_2' y' + v_2 y_2''$$

then $v_1' y' + v_1 y'' + v_2' y' + v_2 y_2'') + b(v_1 y_1' + v_2 y_2') + c(v_1 y_1 + v_2 y_2) = r(x)$

thus $\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = r \end{cases}$ and $v_1' = -\dfrac{r x_2}{W}$, $v_2 = \dfrac{r x_1}{W}$ thus $v_1 = \displaystyle\int -\frac{r x_2}{W}$, $v_2 = \displaystyle\int \frac{r x_1}{W}$

General Coefficients, General Non-Homogenous

Wednesday, November 3, 2021 12:12 PM

Generally: given the equation $y'' + p(x)y' + q(x)y = r(x)$

1) find y_1, y_2: if we know y_1 then we can solve for y_2 using Reduction of Order:

   R of O: given $y_1(x)$, $y_2(x) = v(x)\,y_1(x)$ where $v(x) = \int C \cdot e^{-\int p(x)dx} \cdot \frac{1}{y_1(x)^2}\,dx$

   plug in $y_2(x)$ and solve for $v(x)$ since $y_1(x)$ is known:

   $y_2'' + p y_2' + q y_2 = 0 \rightarrow (v y_1)'' + p(v y_1)' + q(v y_1) = 0$

   $(v'' y_1 + 2v' y_1' + v y_1'') + p(v' y_1 + v y_1') + q(v y_1) = 0$

   $v'' y_1 + 2v' y_1' + p v' y_1 = 0 \longrightarrow y_1 v'' + v'(2y_1' + p y_1) = 0$ let $w = v'$:

   $y_1 w' + (2y_1' + p y_1) w = 0$ solve as a first order equation, $v = \int w$

   soln: $w = C \cdot e^{-\int p(x)\,dx}$, $\frac{1}{y_1(x)^2}$

2) Using variation of parameter, y_1, y_2 we can find the particular soln:

   given y_1, y_2 then $y_p = v_1 y_1 + v_2 y_2$

   then by variation of parameters: $v_1 = \int -\frac{r y_2}{W}$

   $v_2 = \int \frac{r y_1}{W}$

3) The soln is $y = y_p + C_1 y_1 + C_2 y_2$

Cauchy-Euler

Monday, November 1, 2021 2:04 PM

Given the equation: $y'' + p(x)y' + q(x)y = r(x)$

Idea: we can find y_1, y_2 that are basis for the solution space

thus: $y = C_1 y_1 + C_2 y_2 + y_p$

Soln: We can use <u>Cauchy-Euler</u> Equation:

$a x^2 y'' + b x y' + c y = r(x)$ which is an equidimensional equation

if $r(x) = 0$, then $y = 0$ (trivial) and $y = x^m$ where $m \neq 0, 1$

then $a x^2 (m)(m-1)(x^{m-2}) + b x (m)(x^{m-1}) + c(x^m) = 0$ thus

$[am(m-1) + bm + c]x^m = 0 \to am(m-1) + bm + c = 0 \to am^2 + (b-a)m + c = 0$

if m has two distinct roots m_1, m_2:

$y_1 = x^{m_1}$, $y_2 = x^{m_2}$

if m has one distinct root m:

$y_1 = x^m$, $y_2 = x^m \cdot \ln x$

if m has no real roots and complex roots $\alpha + \beta i$:

$y_1 = x^\alpha \cos(\beta \ln x)$ $y_2 = x^\alpha \sin(\beta \ln x)$

Systems of 1st Order Differential Equations

Friday, November 5, 2021 2:34 PM

Idea We can form a system of multiple diff. eqs and solve them:

ex: $\begin{cases} y_1'(x) - y_2'(x) = 1 \\ y_1(x) - y_1'(x) + y_2(x) = 5 \end{cases}$ is a system of first order equations

Def A system of 1^{st} order linear equations are the same as a higher order linear equation:

ex: $y'' - 2y' + 2y = 0$ can be reduced to the system:

let $z = y'$ then $z' - 2z + 2y = 0$ and $z - y' = 0$

thus $\begin{cases} z' - 2z + 2y = 0 \\ z - y' = 0 \end{cases}$ solves the 2^{nd} order equation

ex: $y''' - y'' + y' - y = 3$ can be reduced to the system:

let $z = y'$ $w = y''$ then $w' - w + z - y = 3$, $z - y' = 0$, $w - z' = 0$

thus $\begin{cases} w' - w + z - y = 3 \\ z - y' = 0 \\ w - z' = 0 \end{cases}$ solves the 3^{rd} order equation

Solving System of Linear Equations

Friday, November 5, 2021 2:46 PM

<u>Def</u> Given a system of linear diff eqs we can:

1) rewrite equations so that derivatives are on the left side that there are no derivatives are on the right side this is called the <u>standard form</u>

2) when we have the terms: $y' = ry$ then $y = e^{rx}$

3) when we have an equation that are not independent: use a <u>Matrix!</u>

given the equation: $y^n + ay^{n-1} + by^{n-2} + \dots + cy' + dy + e = 0$

and let $x_1 = y$, $x_2 = y'$, $x_3 = y''$ etc.

then the system:

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 0 \\ -d & -c & & \dots & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -e \end{bmatrix}$$ solves the equation

thus the system is: $\vec{x}'(t) = A\,\vec{x}(t) + \vec{b}$

Review: Matrix Inverse, Determinant

Wednesday, November 10, 2021 4:44 PM

Def Given a $n \times n$ matrix A: we can find A^{-1} by solving:

$$[A \mid I] \sim [I \mid A^{-1}]$$

Def Given a $n \times n$ matrix the determinant of A $\det(A)$:

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$

if $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ then $\det(A) = a \begin{vmatrix} e & h \\ f & i \end{vmatrix} - b \begin{vmatrix} d & g \\ f & i \end{vmatrix} + c \begin{vmatrix} d & g \\ e & h \end{vmatrix} - d \begin{vmatrix} b & h \\ c & i \end{vmatrix}$

where the next order determinants are the minors of the matrix

Note The determinant is the space formed by the vectors.

Def Given the $\det(A)$; if $\det(A) = 0$ then the column vectors co-linear, co-planar, etc. Thus, the column vectors are not Linearly Independent

Review: Linear Independence

Wednesday, November 10, 2021 5:01 PM

Def Three vectors $\vec{u}, \vec{v}, \vec{w}$ are:

Linearly dependent if: $\vec{w} = a\vec{u} + b\vec{v}$

Linearly independent if: no vectors can be written as a linear combination of the others

Def Given $\vec{V_1}, \vec{V_2}, \vec{V_3} \dots$ then $c_1\vec{V_1} + c_2\vec{V_2} + c_3\vec{V_3} + \dots = 0$

if $c_1, c_2, c_3, \dots = 0$, then $\vec{V_1}, \vec{V_2}, \vec{V_3} \dots$ are independent

if $c_1, c_2, c_3 \dots \neq 0$, then $\vec{V_1}, \vec{V_2}, \vec{V_3} \dots$ are dependent

Thm $V_1, V_2, V_3 \dots$ are linearly independent iff $\det([V_1, V_2, V_3, \dots]) \neq 0$

this is called the Wronskian

but if wronskian $= 0$, this does not imply linearly dependence

Review: Eigenvalues, Eigenvectors

Friday, November 19, 2021 3:22 PM

<u>Def</u> Given a an $n \times n$ matrix A:

$\det(A - rI) = 0$ solves for r, the eigenvalues

$(A - rI)\vec{x} = 0$ solves for $\vec{x}$, the eigenvector corresponding to r

<u>Note</u> Since A is implied to be non-invertible, then $A - rI$ must have free variables and thus $\vec{x}$ involves free variables

Solutions to Systems of Linear Diff Eqs

Wednesday, November 10, 2021 5:25 PM

Def Given a system of 1st order linear diff eqs:

example: $x_1' = x_1$ $\quad x_1 = c_1 e^{x}$

$x_2' = 2x_2$ $\quad x_2 = c_2 e^{2x}$

then the solution might be: $\begin{bmatrix} c_1 e^{t} \\ c_2 e^{2t} \end{bmatrix} = c_1 \begin{bmatrix} e^{t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$

and we can treat $\vec{x}_1 = \begin{bmatrix} e^{t} \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$ so the solution is $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

and $\vec{x}_1$, $\vec{x}_2$ are the fundamental solutions

Def In general if $\vec{x}' = A\vec{x}$ if A is $n \times n$ then we can find n vectors:

$\vec{x}_1, \vec{x}_2, \dots \vec{x}_n$ such that each vector is a solution to the system

and thus the solution $\boxed{\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_2}$

Idea We must always check if $\vec{x}_1, \vec{x}_2, \dots \vec{x}_n$ are LI by taking the $\det([\vec{x}_1, \vec{x}_2, \dots \vec{x}_n])$

Def In general if $\vec{x}' = A\vec{x} + \vec{b}$ then $\vec{x} = \vec{b}$ gives the particular solution $\vec{x}_p$

the final solution is: $\vec{x} = c_1 \vec{x}_1 + \dots + c_n \vec{x}_n + \vec{x}_p$

Def The fundamental matrix for a system with solutions $\vec{x}_1 \dots \vec{x}_n$ is:

$X = [\vec{x}_1, \dots \vec{x}_n]$

Solving Homogenous Constant Systems

Wednesday, November 17, 2021 12:03 PM

Def Given the system: $\vec{x}' = A\vec{x}$ that is homogenous and has constant coefficients:

if $A\vec{u} = r\vec{u}$ then $\vec{x} = e^{rt} \cdot \vec{u}$

where r is the eigenvalue and $\vec{u}$ is the corresponding eigenvector.

Idea We can find $r, \vec{u}$ by finding the eigenvalue, eigenvector of A

Def If r is real:

$$\vec{x} = C_n e^{rt} \cdot \vec{u}$$

Def If r is complex in the form $\alpha \pm \beta i$, and $z = \vec{a} \pm \vec{b} i$ is the corresponding eigenvectors

$$\vec{x} = C_n \left[e^{\alpha t} \cos\beta t \cdot \vec{a} - e^{\alpha t} \sin\beta t \cdot \vec{b} \right] + C_n \left[e^{\alpha t} \sin\beta \cdot \vec{a} + e^{\alpha t} \cos\beta \cdot \vec{b} \right]$$

Solving Nonhomogenous Constant Systems

Monday, November 22, 2021 4:29 PM

Def Given the system $\vec{x}' = A\vec{x} + \vec{b}$

we can still use Reduction of Order, Variation of Parameters

Laplace Transform

Sunday, November 28, 2021 11:07 PM

Def Given some function $f(t)$ then the Laplace Transform:

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$$ and maps function of $t \to$ function of s

and inverse-Laplace transform:

$$\mathcal{L}^{-1}\{s\}(t) =$$

Ex given $f(t) = 1$:

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} dt = \lim_{A \to \infty} \left(-\frac{1}{s} e^{-st}\right)\Big|_{t=0}^{t=A}$$

$$= \lim_{A \to \infty} \left(-\frac{1}{s} e^{-st} + \frac{1}{s}\right) = \lim_{A \to \infty} \left(-\frac{1}{s} e^{-st}\right) + \lim_{A \to \infty} \left(\frac{1}{s}\right)$$

when $s > 0$, $\lim \to 0$ $\quad = \frac{1}{s}$

when $s \leq 0$, $\lim \to \infty$

thus $\mathcal{L}\{1\}(s) = \frac{1}{s}$ for $s > 0$

Idea We can solve an equation $y''(t) + y'(t) + y(t) = 0$

$$\mathcal{L}\{\underbrace{y''(t) + y'(t) + y(t)}_{\text{derivatives will dissappear}}\}(s) = \mathcal{L}\{0\}(s)$$

Basic Laplace Transforms

Sunday, November 28, 2021 11:27 PM

Def Basic Laplace Transforms:

$\mathcal{L}[\,1\,] = \frac{1}{s}$ for $s > 0$

$\mathcal{L}[\,e^t\,] = \frac{1}{1-s}$ for $s > 1$

$\mathcal{L}[\,e^{at}\,] = \frac{1}{s-a}$ for $s > a$

$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$ for $s > 0$

$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}$ for $s > 0$

$\mathcal{L}[y'] = s \cdot \mathcal{L}[y] - y(0)$

$\mathcal{L}[y''] = s \cdot \mathcal{L}[y'] - y'(0) = s^2 \cdot \mathcal{L}[y] - s \cdot y(0) - y'(0)$

$\mathcal{L}[y^n] = s^n \cdot \mathcal{L}[y] - \sum_{i=0}^{n-1} s^{(n-1-i)}\, y^i(0)$

Def Linearity Property: $\mathcal{L}[f(t) + g(t)](s) = \mathcal{L}[f(t)](s) + \mathcal{L}[g(t)](s)$

and : $\mathcal{L}[c \cdot f(t)](s) = c \cdot \mathcal{L}[f(t)](s)$

Def Shifting Property: $\mathcal{L}[e^{at} \cdot f(t)](s) = \mathcal{L}[f(t)](s-a) = F(s-a)$

Note We will denote the Laplace Transform $\mathcal{L}\{f(t)\}(s)$ as $F(s)$

Solving Differential Equations With Laplace Transforms

Sunday, November 28, 2021 11:24 PM

<u>Def</u> Given some differential equation we can:

1) Perform Laplace Transform $t \to s$
2) Solve the transformed equation
3) Perform inverse Laplace Transform $s \to t$

<u>Ex</u> Given the equation $y' + ay = b$:

$$\mathcal{L}[y' + ay](s) = \mathcal{L}[b](s)$$

$$\mathcal{L}[y'](s) + a\mathcal{L}[y](s) = b\mathcal{L}[1](s)$$

$$sY(s) - y(0) + aY(s) = \frac{b}{s}$$

$$(s+a)Y(s) = \frac{b}{s} + y(0)$$

$$Y(s) = \left(\frac{1}{s+a}\right)\left(\frac{b}{s} + y(0)\right)$$

thus: $y(t) = \mathcal{L}^{-1}\left[\left(\frac{1}{s+a}\right)\left(\frac{b}{s} + y(0)\right)\right]$