Vectors Quick Notes

Monday, October 5, 2020 4:19 PM

Vectors

$\langle x, y, z \rangle$

dot product : sum of product of dims.

$a \cdot a = |a|^2$

inner product $\longleftrightarrow$ dot product.

$\measuredangle$ between 2 vectors

$a \cdot b = |a||b| \cos\theta$

$\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$

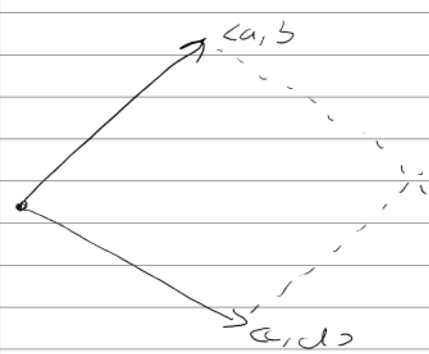
$|a \cdot b| \leq |a||b|$

projection of v on a

$= \frac{a \cdot v}{|a|^2} \cdot a$

determinant

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\langle a, b \rangle$ $\langle c, d \rangle$



$\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$

volume of the parallelpiped

generalized volume

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

gaussian elimination:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{vmatrix}$$

- row swap
- multiply a row
- add a row

$$L_2 + \frac{3}{2} L_1 \to L_2$$

$$\begin{vmatrix} i & j & k \\ \langle x, y, z \rangle \\ \langle x, y, z \rangle \end{vmatrix} \qquad \begin{vmatrix} 2 & 1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -2 & 1 & 2 \end{vmatrix}$$

$$L_3 + L_1 \to L_2$$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{vmatrix} \qquad \begin{vmatrix} 2 & 1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 & 1 \end{vmatrix}$$

$$L_3 - 4L_1 \to L_3$$

$$\begin{vmatrix} 2 & 1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 5$$

Quick cross product

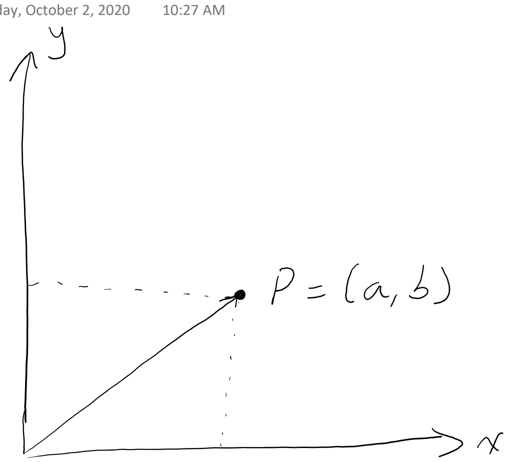
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$$aei + bfg + cdh - (ceg + afh + bdi)$$

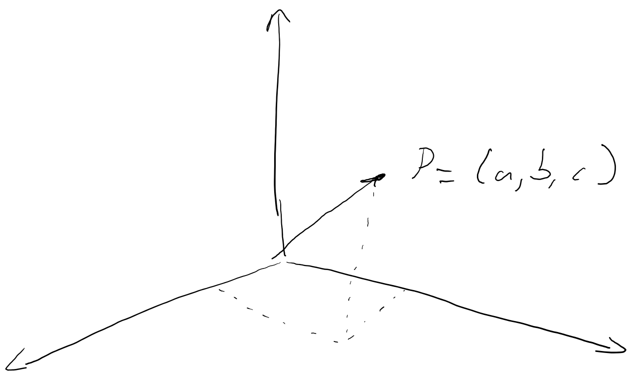
Vectors in 2D and 3D

Friday, October 2, 2020 10:27 AM



every point in $\mathbb{R}^2$ has 2 coordinates, (x, y)

you can use any point to make a vector $\vec{v}$ by drawing an arrow from the origin

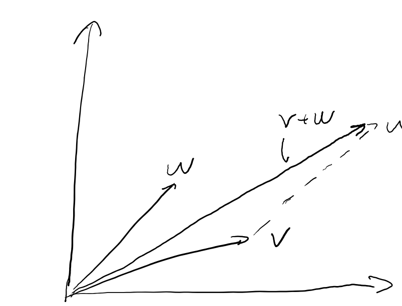


in $\mathbb{R}^3$, there are 3 coordinates $\langle x, y, z \rangle$

Notation

The set of real numbers is $\mathbb{R}$. The set of pairs of real numbers (2D vectors) is denoted by $\mathbb{R}^2$. 3D is $\mathbb{R}^3$

Adding Vectors



Geometric Version

$\vec{w} = (1, 2, 3)$ $\quad$ $\vec{v} + \vec{w} = (1+2, 2+8, 3+1)$

$\vec{v} = (2, 8, 1)$ $\quad$ $= (3, 10, 4)$

$\boxed{\text{add elementwise}}$

comment. - can't add 2D & 3D vectors together

Multiplying by a Scalar

start with $\vec{v} \in \mathbb{R}^3$

(notation $\vec{v} \in \mathbb{R} = \vec{v}$ is in $\mathbb{R}^3$ aka is a 3D vector)

and $\lambda \in \mathbb{R}$ (a real #)

lambda (just a variable)

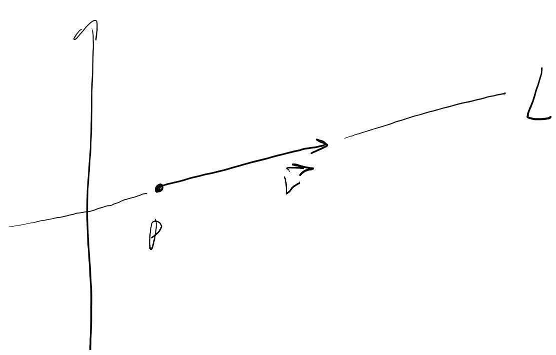
$\lambda \cdot \vec{v}$ is the elementwise product of $\vec{v}$ by λ

eg. if $\vec{v} = (a, b, c)$

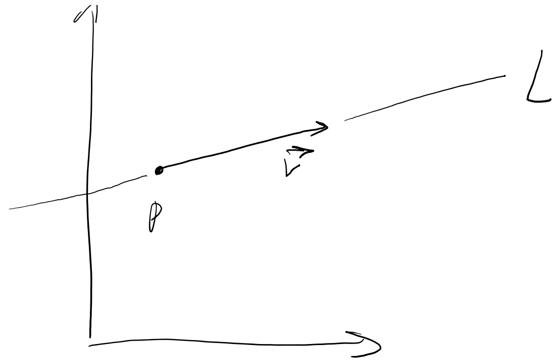
$\lambda \cdot \vec{v} = (\lambda a, \lambda b, \lambda c)$

Geometrically we are lengthening or shortening a vector (negatives reflect it backwards)

Lines in 2D & 3D

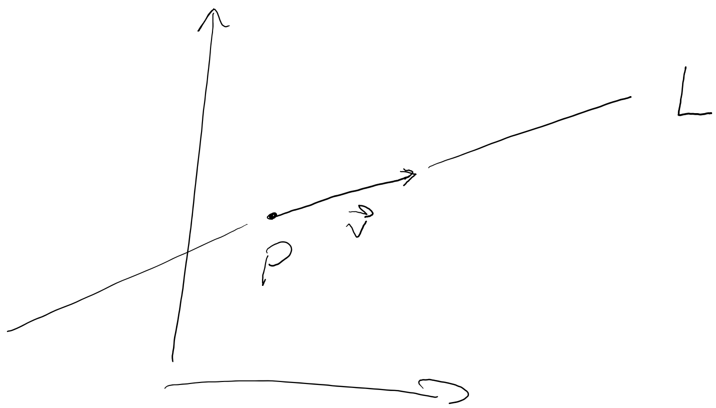


Determined by a point on L P, and a direction $\vec{v}$



Determined by a point on L P, and a direction $\vec{v}$

Parameterization



$P = (a,b)$

$\vec{v} = (c,d)$

Q can be found by starting at P and moving in the direction of $\vec{v}$ (or exact opposite) aka the scaler multiple.

thus $Q = P + \lambda \cdot \vec{v}$

Parametric equation for L:

$$L(t) = P + t \cdot \vec{v}$$

$$= (a,b) + t(c,d)$$

$$= (a+ct, b+dt)$$

Excercise:

1) Parameterize $\overline{L}$ starting at

$(1,2,3)$ pointing $(1,-1,0)$

$$L(t) = (1+t,\ 2-t,\ 3)$$

2) Parameterize line between $(4,5)$ and $(2,-2)$

$$L(t) = (4-2t,\ 5-7t)$$

or $L(t) = (2+2t,\ -2+7t)$

Inner Product, Length, Distance

Monday, October 5, 2020 9:53 AM

There are many ways to define vector multiplication

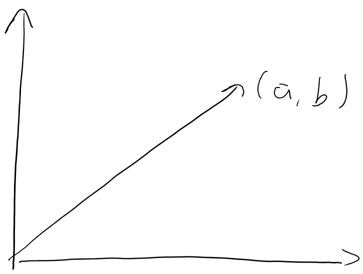
Scalar multiplication takes 1 scalar and 1 vector and gives a vector

Inner Product takes 2 vectors ($\mathbb{R}^2$ or $\mathbb{R}^3$) and gives a $\mathbb{R}$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2)$$
$$= x_1 x_2 + y_1 y_2 + z_1 z_2$$

Geometric meaning:

Let $\vec{v} = (a, b) \in \mathbb{R}^2$



$$\|\vec{v}\| = \sqrt{a^2 + b^2}$$
$$= \sqrt{\vec{v} \cdot \vec{v}}$$

So the length of the vector is $\sqrt{\vec{v} \cdot \vec{v}}$

Distance between 2 points:

$P = (1, 2, 4)$ $\quad Q = (0, 1, 1)$

$$\vec{v} = P - Q = (1, 1, 3)$$
$$d = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{1 + 1 + 9} = \sqrt{11}$$

Some Properties:

Let $\lambda = \mathbb{R}$ $\quad \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ or $\mathbb{R}^3$

A) $(\lambda \vec{v}) \cdot \vec{w} = \lambda (\vec{v} \cdot \vec{w})$ commutes with scalar

B) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ distributive

C) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ reflexive

If 2 vectors point in the same direction:

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\|$

For all 2 vectors:

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$$

So geometrically, the inner product tries to find the product of lengths, and the $\cos\theta$ corrects for the difference in directions.

Consequences:

$\vec{v}$ and $\vec{w}$ are orthogonal ($\perp$) if $\vec{v} \cdot \vec{w} = 0$

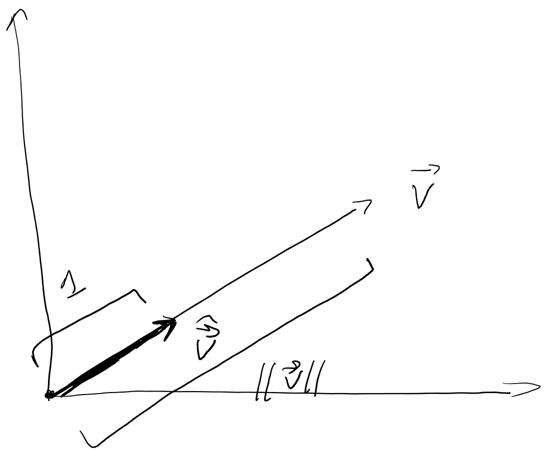
Unit Vectors

Wednesday, October 7, 2020 10:07 AM

Terminology

$\hat{i}, \hat{j}, \hat{k}$ represent the unit vectors along the x, y, z axes

other unit vectors are any vector with magnitude of 1, also called normalized vector

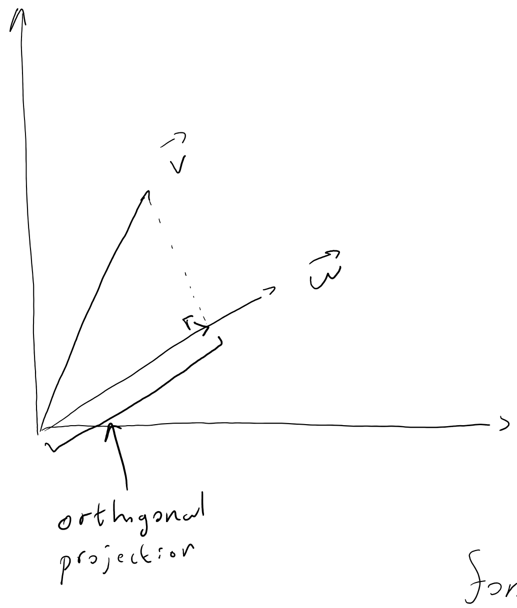


Normalizing a vector gives a vector in the same direction but with length of 1

$$\hat{\vec{v}} = \frac{\vec{v}}{\|\vec{v}\|}$$ normalizing vector v

Orthogonal Projection

Friday, October 9, 2020 10:04 AM



for any vector $\vec{v}$, there is a scalar multiple of $\vec{w}$ such that

$$\vec{v} - \lambda\vec{w}$$

is orthogonal to $\vec{w}$

and where $\lambda \cdot \vec{w}$ is the orthogonal projection of $\vec{v}$ onto $\vec{w}$

formula

$$(\vec{v} - \lambda\vec{w}) \cdot \vec{w} = 0$$

$$\vec{v} \cdot \vec{w} - \lambda\, \vec{w} \cdot \vec{w} = 0$$

$$\vec{v} \cdot \vec{w} = \lambda(\vec{w} \cdot \vec{w})$$

$$\lambda = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}$$

$$\lambda \cdot \vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \cdot \vec{w}$$

Cross Product

Friday, October 16, 2020 10:03 AM

Cross Product

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ < & \vec{v} & > \\ < & \vec{w} & > \end{vmatrix}$$

a 3×3 determinant with the 3 unit vectors at the top and $\vec{v}$ and $\vec{w}$ as row vectors underneath

$$\| \vec{v} \times \vec{w} \| = \| \vec{v} \| \times \| \vec{w} \| \sin \theta$$

where θ is the $\angle$ between $\vec{v}$ and $\vec{w}$

Properties:

Distributive: $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$

Skew Symetric: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

Scalar : $\lambda \vec{u} \times \vec{v} = \lambda (\vec{u} \times \vec{v})$

Triple Product : $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$

Matrices and Determinants

Friday, October 9, 2020 10:24 AM

A 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ where a_{ij} is elements by i row and j column

The determinant is a real number:

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

A 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

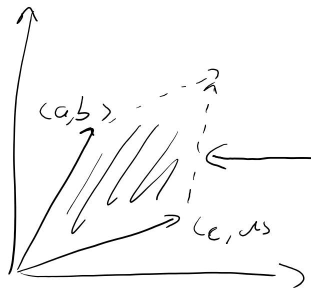
= sum of the products of the backwards diagonals minus the sum of the products of the backwards diagonals

Geometric Meaning:

2×2: $\begin{vmatrix} \langle a & b \rangle \\ \langle c & d \rangle \end{vmatrix}$

$| a \; b |$

$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = area of parallelogram



3×3 : volume of the parallelopiped.

Right Hand Rule

if $\begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix}$ follow the right hand rule, then $\begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix} \geq 0$, otherwise it is ≤ 0

Planes

Friday, October 16, 2020 10:27 AM

Equation:

Point p that is on the plane $= (x_0, y_0, z_0)$

Vector $\vec{v}$ that is normal to the plane (t) $= \langle a, b, c \rangle$

(x, y, z) is in the plane if

$$\vec{v} \cdot ((x, y, z) - p)$$

$$= a \cdot (x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Distance from point to plane:

is the projection of the distance vector from the origin to the point onto the normal vector of the plane as it goes to 0.

Thinking of Vectors as Matricies

Saturday, October 17, 2020 5:36 PM

$$\vec{v} = \langle x, y, z \rangle = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\vec{v}^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

Visualization: Level Sets and Taking Sections

Monday, October 19, 2020 10:24 AM

Level Sets

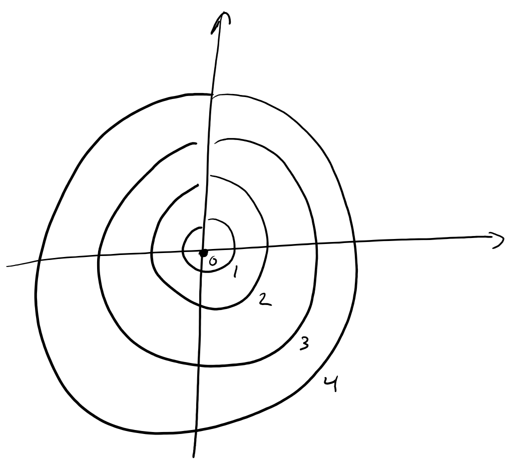
For some $f(x, y)$, we can graph a representation by assigning $f(x, y) = c$ for some set of c.

Creates a ~~to~~pological graph.

Example:

$z = x^2 + y^2$

for $z = c \in [0, 4]$



Taking Sections

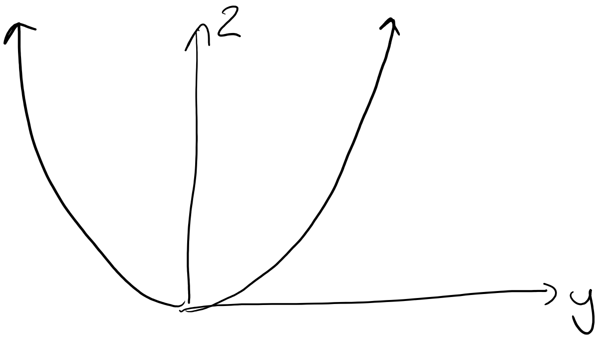
Set axial values to 0, then graph in 2D space.

$f(x, y) = x^2 + y^2$

$2 = x^2 + y^2$

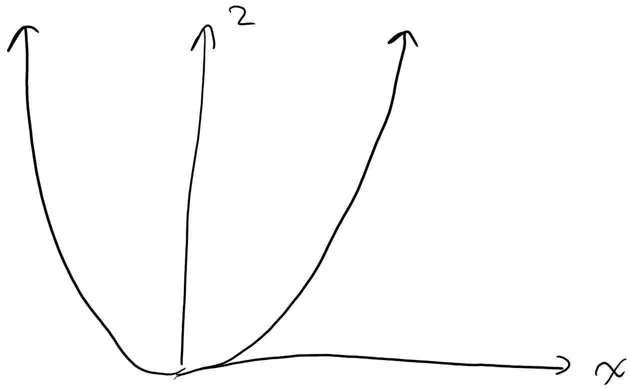
$z = 0 + y^2$

$z = y^2$ @ $x = 0$



$z = x^2 + 0$

$z = x^2$ @ $y = 0$



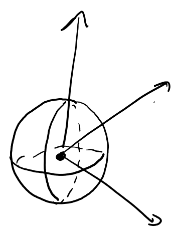
For $\mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z) = x^2 + y^2 + z^2$

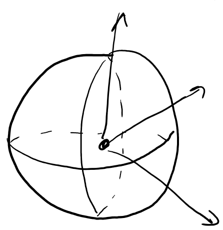
we must take level sets to visualize

ie:

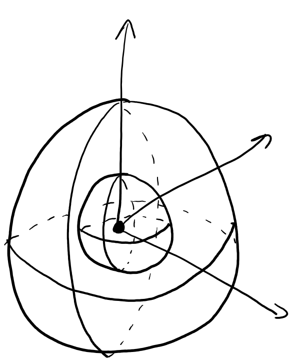
$1 = x^2 + y^2 + z^2 \ldots =$



$2 = x^2 + y^2 + z^2 \ldots =$



so



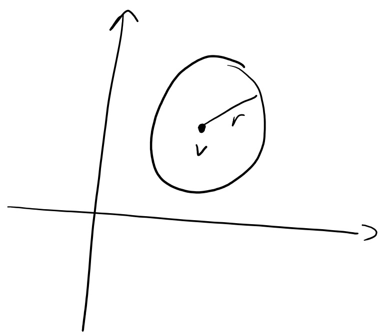
in this way we can visualize functions in more complex spaces.

Sets and Neighborhoods

Wednesday, October 21, 2020 10:16 AM

Sets

Closed set in $\mathbb{R}^2$ or $\mathbb{R}^3$



all points radius r from vector $\vec{v}$

Open set

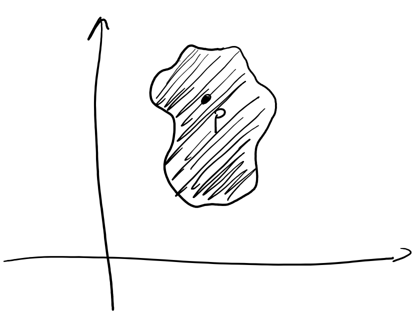
a closed set but without the boundary

Boundary set

the boundary points

Neighborhoods

Similar to a set but with no radial definition



also follows the closed, open, boundary rules.

Limit of P

Limit of $f(x,y)$ as (x,y) approaches P is

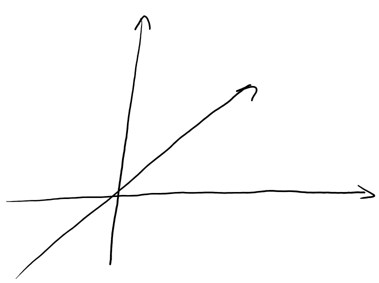
line L. $\lim_{(x,y)\to p} f(x,y) = L$

there is an open neighborhood of p which maps into this interval.

A function is continous if for every $(a,b) \in \mathbb{R}^2$, the limit of $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Example of function w/ no limit as $(x,y) \to (0,0)$

$$f(x,y) = \frac{x^2}{x^2+y^2}$$



If you approach along the line $y = ax$

$$f(x,ax) = \frac{x^2}{x^2+a^2x^2} = \frac{1}{1+a^2}$$

therefore $f(x,ax)$ is constant but changes

This means the $\lim_{(x,y)\to(a,b)} \frac{x^2}{x^2+y^2}$ does not exist

Differentiation

Differentiable functions are the ones that can be approximated by linear functions.

Linear function:

$$f(x, y) = \mathbb{R}^2 \to \mathbb{R}$$
$$= a + bx + cy$$
$$f(x, y, z) = \mathbb{R}^3 \to \mathbb{R}$$
$$= a + bx + cy + dz$$

$$(x_0, y_0) \in \mathbb{R}^2$$

other example: $a + b(x - x_0) + c(y - y_0)$

A function is differentiable if at $(x_0, y_0) \in \mathbb{R}^2$, if there is a linear function $l(x, y) = a + bx + cy$ so that

$$\lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - l(x, y)}{\|(x, y) - (x_0, y_0)\|} = 0$$

$$\lim_{(x,y,z) \to (x_0, y_0, z_0)} \frac{f(x, y, z) - l(x, y, z)}{\|(x, y, z) - (x_0, y_0, z_0)\|} = 0$$

We say that l is the linear approximation of f at $(x_0, y_0)(z_0)$

A consequence of this is $l(x_0, y_0) = f(x_0, y_0)$

We can use this to define partial derivatives:

the partial derivative $\frac{\partial f}{\partial x}(x_0, y_0) = b$

$\frac{\partial f}{\partial x}(x_0, y_0) = c$

Computing Partial Derivatives

If $f(x, y)$ is differentiable at (x_0, y_0) then:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Note: that $f(x_0 + h, y_0)$ is a 1-variable function in h, and $\frac{\partial f}{\partial x}(x_0, y_0) = \left. \frac{\partial f(x_0 + h, y_0)}{\partial h} \right|_{h=0}$

Linear Approximation

The linear approximation of some function $f(x, y)$ at (x_0, y_0):

$$l(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (\langle x, y \rangle - \langle x_0, y_0 \rangle)$$

or:

$$l(x, y) = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + f(x_0, y_0)$$

and for $g(x, y, z)$ at (x_0, y_0, z_0):

$$l(x, y, z) = g(x_0, y_0) + \nabla g(x_0, y_0, z_0) \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle)$$

or:

$$l(x, y) = \frac{\partial g}{\partial x}(x - x_0) + \frac{\partial g}{\partial y}(y - y_0) + \frac{\partial g}{\partial z}(z - z_0) + g(x_0, y_0, z_0)$$

Example

$f(x,y) = e^{x \cdot y}$

$\frac{\partial f}{\partial x}(x,y) = y e^{x \cdot y}$

$\frac{\partial f}{\partial y}(2,1) = x e^{x \cdot y} \Big|_{(2,1)} = 2 \cdot e^2$

$g(x,y,z) = \sin(xz) + y^2 z$

$\frac{\partial g}{\partial z} = x\cos(xz) + y^2 \Big|_{(1,1,1)}$

$g(x,y,z) = 1 + x^2 + xy + yz$ $\qquad 1 + 1^2 + 1 \cdot 1 + 1 \cdot 1 = 4$

$\frac{\partial g}{\partial x} = 2x + y = 2 + 1 = 3$

$\frac{\partial g}{\partial y} = x + z = 1 + 1 = 2$

$\frac{\partial g}{\partial z} = y = 1$

$l = a + 3x + 2y + z$ $\qquad l(1,1,1) = g(1,1,1)$

$a + 3 + 2 + 1 = 4$

$a + 6 = 4$

$a = -2$

$l = -2 + 3x + 2y + z$

alternatively you can use "point - slope" form:

$$\ell = 3(x-1) + 2(y-1) + (z-1) + 4$$

Gradient

Monday, October 26, 2020 10:01 AM

Gradient

Let $f(x,y)$ (or $g(x,y,z)$) be a differentiable function,

the derivative of $f(x,y)$ at (x_0, y_0) is:

$$Df(x_0,y_0) = \left[\frac{\partial f}{\partial x}(x_0,y_0) \quad \frac{\partial f}{\partial y}(x_0,y_0) \right]$$

The gradient of $f(x,y)$ at (x_0,y_0) is the vector

$$\nabla f(x_0,y_0) = \left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right\rangle$$

for $g(x,y,z)$:

$$Dg(x_0,y_0,z_0) = \left[\frac{\partial g}{\partial x}(x_0,y_0,z_0) \quad \frac{\partial g}{\partial y}(x_0,y_0,z_0) \quad \frac{\partial g}{\partial z}(x_0,y_0,z_0) \right]$$

$$\nabla g(x_0,y_0,z_0) = \left\langle \frac{\partial g}{\partial x}(x_0,y_0,z_0), \frac{\partial g}{\partial y}(x_0,y_0,z_0), \frac{\partial g}{\partial z}(x_0,y_0,z_0) \right\rangle$$

In general, the derivative is the matrix of each partial derivative, and the gradient is the vector where corresponding dimensions are the partial derivatives.

Geometrically

$\nabla f(x_0,y_0,z_0)$

1. points in the direction of (x_0,y_0,z_0)
2. $\|\nabla f\|$ is 1.

If $f(x,y,z) = c$

then ∇f is tangent to the level set at c

and the tangent plane is

$$0 = \nabla f \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

∇f points in the direction of the greatest increase.

Directional Differentiation

Wednesday, November 4, 2020 9:59 AM

Finds the rate of change of a function towards a given direction (and velocity).

$$D_v f = \nabla f \cdot \text{unit}(v)$$

and should return a scalar value.

Second Derivative

Friday, November 6, 2020 10:32 AM

Suppose $f(x, y)$ is a C^2- function (C^2 = twice continously differentiable). The second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Hessian Matrix

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

If f is C^2 - function, then $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Use

Determining wether a critical point is a maximum or minimum where a critical point is where $\nabla f(x, y) = \langle 0, 0 \rangle$.

Let $f(x,y)$ be a function with a critical point (x_0, y_0),
let the Hessian matrix

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ be the matrix at (x_0, y_0)

The second derivative test (for 2D functions):

if $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^2 f}{\partial y^2} > 0$ AND $\det(H) > 0$ then (x_0, y_0) is a local min

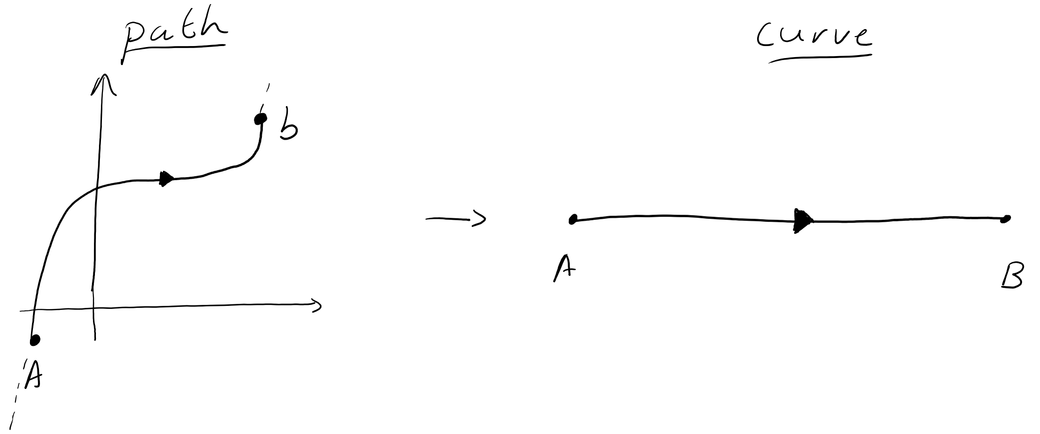
if $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^2 f}{\partial y^2} < 0$ AND $\det(H) > 0$ then (x_0, y_0) is a local max

if $\det(H) < 0$ then (x_0, y_0) is a saddle point.

Paths & Curves

A path is a function $\bar{c}(t) = [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$)

A curve is the range of a path.



The derivative of a path $\bar{c}(t) = (x(t), y(t))$ is the vector

$$\bar{c}'(t) = \langle x'(t), y'(t) \rangle$$

the meaning of $\bar{c}'(t)$:
if $\bar{c}'(t) \neq \langle 0, 0 \rangle$ then it's a vector tangent to the path

The velocity of a path is $\bar{c}'(t) = \langle \frac{\partial x}{\partial t} \quad \frac{\partial y}{\partial t} \quad \frac{\partial z}{\partial t} \rangle$

The acceleration of a path is $\bar{c}''(t) = \langle \frac{\partial^2 x}{\partial t^2} \quad \frac{\partial^2 y}{\partial t^2} \quad \frac{\partial^2 z}{\partial t^2} \rangle$

Force on an object is $m \cdot \bar{c}''(t)$.

Displacement and Arc Length

Monday, November 23, 2020 10:36 AM

For a curve $C \in \mathbb{R}^2$ or $\mathbb{R}^3$ parameterized by the path $c(t) = \langle x(t) \quad y(t) \quad z(t) \rangle$

1. The displacement between $t = a$ and $t = b$ is $c(b) - c(a)$

2. The length of the arc between $t = a$ and $t = b$:

$$\Delta x = x(b) - x(a) = \int_a^b x'(t)\,dt$$

$$\Delta y = y(b) - y(a) = \int_a^b y'(t)\,dt$$

$$\Delta z = z(b) - z(a) = \int_a^b z'(t)\,dt$$

$$\text{Displacement} = \| \langle \Delta x \quad \Delta y \quad \Delta z \rangle \| = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

3. The arc length is the length of the curve C between $t = a$ and $t = b$. Rather than integrating velocity, we integrate speed.

$$\bar{c}(t_{i+1}) - \bar{c}(t_i) = \text{displacement} \approx \bar{c}\,'(t_i) \cdot \Delta t$$

$$\text{length between } \bar{c}(t_i) \text{ and } \bar{c}(t_{i+1}) \approx \| \bar{c}(t_{i+1}) - \bar{c}(t_i) \| \approx \| \bar{c}\,'(t_i) \| \Delta t$$

So the arc length $\approx \sum^{n} \| \bar{c}\,'(t_i) \| \Delta t$

So the arc length $\approx \sum_{i=1}^{n} \| \bar{c}'(t_i) \| \Delta t$

as $n \to \infty$

Arc length of $\bar{c}$ $= \displaystyle\int_a^b \| \bar{c}'(t) \| \, dt$

note: This formula works in any dimmensionality.

sometimes the arc length and the length of the curve.

eg. $\bar{c} : [0, 4\pi] \to \mathbb{R}^2$

$t \longrightarrow (\cos(t), \sin(t))$

$$\text{arclength} = \int_0^{4\pi} \| c'(t) \| \, dt$$

$$= \int_0^{4\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2} \, dt$$

$$= \int_0^{4\pi} \sqrt{1} \, dt$$

$$= \boxed{4\pi}$$

but the length of the path should be 2π.

in this example, we go around the circle twice.

Rules of Differentiation

Wednesday, October 28, 2020 10:46 AM

Rules

A) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Let $\lambda \in \mathbb{R}$

$$D(\lambda f(x,y)) = \lambda Df(x,y)$$

$$\begin{bmatrix} \cancel{\frac{\partial \lambda f}{\partial x}} & \frac{\partial \lambda f}{\partial y} \end{bmatrix} = \lambda \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

also: $\nabla(\lambda f) = \lambda \nabla f$

B) $f, g : \mathbb{R}^2 \to \mathbb{R}$

$$D(f+g) = Df + Dg$$

$$[a \quad b] + [c \quad d] = [a+c \quad b+d]$$

C) Product Rule

$f, g : \mathbb{R}^2 \to \mathbb{R}$

$$D(fg) = f Dg + g Df$$

D) Quotient Rule

$f, g : \mathbb{R}^2 \to \mathbb{R}$

$$D\left(\frac{f}{g}\right) = \frac{g\,Df - f\,Dg}{g^2}$$

E) Chain Rule

The easy case

Let $\bar{c} : \mathbb{R} \to \mathbb{R}^3$ be a curve

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar

$$= \underbrace{D(f \circ \bar{c})(t_0)}_{\text{real \#}} = \underbrace{\nabla f(\bar{c}(t_0))}_{\text{3D vector}} \cdot \underbrace{D\bar{c}(t_0)}_{\text{3D vector}} \quad \text{(Chain Rule)}$$

where $\nabla f = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right)$ takes inputs from the vector output of $\vec{c}(t_0)$.

The hard case:

Let $F : \mathbb{R}^{\ell} \to \mathbb{R}^{m}$ so $F(\langle x_1, \dots x_\ell\rangle) = \langle F_1(\langle x_1, \dots x_\ell\rangle) \dots F_m(\langle x_1, \dots x_\ell\rangle)\rangle$

$G : \mathbb{R}^{m} \to \mathbb{R}^{n}$ so $G(\langle x_1 \dots x_m\rangle) = \langle G_1(\langle x_1, \dots x_m\rangle) \dots G_n(\langle x_1, \dots x_m\rangle)\rangle$

We want to find $D(G \circ F)$ (notice that $R^{\ell} \to R^{m} \to R^{n}$ must be followed.

$$\underbrace{D(G \circ F)}_{n \times \ell} = \underbrace{DG(F(x_1, \dots x_\ell))}_{n \times m} \cdot \underbrace{DF(x_1, \dots x_\ell)}_{m \times \ell}$$

matrix multiplication

Where

$DH(x_1, \dots x_n) = (y_1 \dots y_m)$

$$= \begin{array}{c c} & \begin{array}{ccc} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{array} \\ \begin{array}{c} y_1 \\ \vdots \\ y_m \end{array} & \left[\begin{array}{ccc} \quad & \quad & \quad \\ & & \\ & & \end{array} \right] \end{array}$$

Matrix Multiplication

Friday, October 30, 2020 10:28 AM

Given $A = [\quad]_{n \times m}$

$B = [\quad]_{m \times l}$

$$A \times B = [\quad]_{n \times l}$$

where the term T_{ij} is computed by taking the ith row vector of A and $\times$ jth column vector of B

$$(A \times B)_{ij} = A_{\text{row } i} \cdot B_{\text{col } j} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Optimization

Friday, November 13, 2020 10:37 AM

Finding Local Extrema:

Critical points are where $\nabla f = 0$

Use heshian matrix to determine type of critical point.

Finding Global Extrema:

Let $A \in \mathbb{R}^2$ be a closed and bounded set.

closed: is contained around some boundary path

bounded: is finite in size

Let $f : \mathbb{R}^2 \to \mathbb{R}$

To find global extrema of f in the set A

1) Find all the local extrema on the interior of A of f. ie $\nabla f = 0$

2) Find all the local extrema on the boundary of A of f. ie plug A into f and find critical points.

3) The global extrema is the most extreme (largest/smallest value(s) between the local extrema in the set A and the boundary.

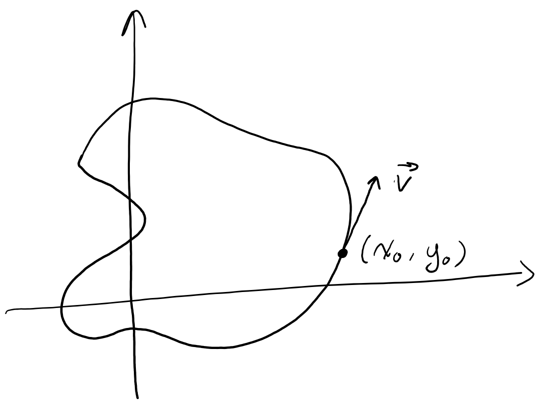
LaGrange Multipliers

Monday, November 16, 2020 10:36 AM

We needed to optimize the functions ∂A (on some boundary)
LaGrange Multipliers help optimize on ∂A, especially when ∂A is a level set

The idea

Suppose we have curve $C \in \mathbb{R}^2$
and $f: \mathbb{R}^2 \to \mathbb{R}$



If $\vec{v}$ is tangent to C at (x_0, y_0) then the directional derivative:

$\nabla f(x_0, y_0) \cdot \vec{v}$

approximates the change in f as you move along $\vec{v}$.

If $\nabla f(x_0, y_0) \cdot \vec{v} > 0$, then f is increasing in direction $\vec{v}$
If $\nabla f(x_0, y_0) \cdot -\vec{v} < 0$, then f is decreasing in direction $-\vec{v}$
both mean that the point is not at a maximum or a minimum of f on C.

The only way (x_0, y_0) can be a local extrema is if $\nabla f(x_0, y_0)$ is normal to curve C.

Equation:

$\nabla f = \lambda \cdot \nabla b$

where ∇b is the gradient of the boundary as a non-parametrized function (ie. $x^2+y^2=4$)

Theorem

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function and let $S \in \mathbb{R}^m$ be the levelset be the levelset of level c.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function

Suppose $(a_1, \dots a_n) \in S$ is a local extremma for f on S.

If $\nabla g(a_1, \dots a_n) \neq 0$ (ie a normal vector)

then $\nabla f(a_1, \dots a_n) = \lambda \nabla g(a_1, \dots a_n)$ for some $\lambda \in \mathbb{R}$

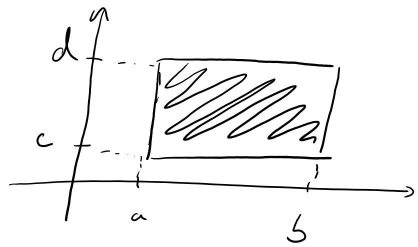
Tl:dr: the critical points for a function f on a level set for g are points where $\nabla f = \lambda \nabla g$ or $\nabla g = 0$

Integration

Monday, November 30, 2020 10:02 AM

Double Integrals as volumes

Let $R = [a,b] \times [c,d] \in \mathbb{R}^2$
be the following rectangle:



The volume over $z = f(x,y)$ between $[a,b] \times [c,d]$
or $\mathbb{R}$ is S:

$0 \leq z \leq f(x,y)$

where $(x,y) \in R)$

$$\iint_R f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

is the volume of S

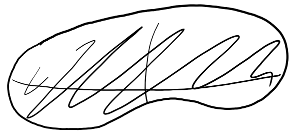
read: "the integral" of f over R

Cavalieri"s Principle

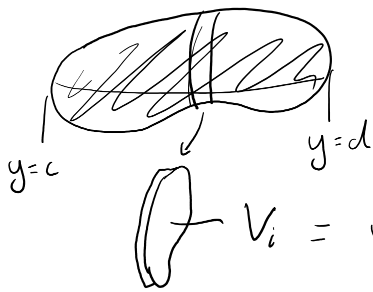
Monday, November 30, 2020 10:16 AM

Cavalieri's Principle

Given some 3D volume:



we can slice the volume into slices



V_i = volume between y_{i-1} x y_i

Volume of $S = \sum_{i=1}^{n} V_i$

where V_i = (cross sectional area) $\cdot \Delta y$

We define a function

$A(y)$ = (cross sectional area at y)

Volume of $S \approx \sum_{i=1}^{n} A(y_i)\Delta y$

as $n \to \infty$

$$S = \int_c^d A(y)\,dy$$

The volume is the integral of the cross sectional area.

Integration Rules

Monday, November 30, 2020 10:31 AM

"Partial Integrals"

$$\int x + y \; dx$$

y is constant, integrat with normal rules

$$= \frac{1}{2}x^2 + yx$$

Iterative Integrals

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy} \Big)\,dx$$

Is the same as:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

- Treat x as constant
- Integrate w/ respect to y
- Computes the x as constant which gives the cross sectional area

Summary

Wednesday, December 2, 2020 10:12 AM

Integrating over rectangles

Let $R = [a, b] \times [c, d] \in \mathbb{R}^2$

We want to define

$$\iint_R f(x, y)\,dy\,dx$$

for any continous function $f(x, y)$

Rieman Sum Definition

The integral of $f(x,y)$ over R is:

$$\iint_R f(x,y)\,dy\,dx = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_i)\,\Delta y\,\Delta x \right)$$

Properties

1. If $f(x,y) \geq 0$ then:

   Riemann Sum = Volume of Solid

2. $\iint_R f(x,y)\,dy\,dx$ can be computed iteratively

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Integrating over general regions

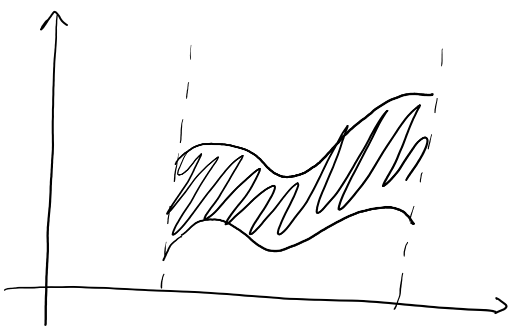
- I 1D, the only regions we integrate over are intervals.
- In 2D, there are non rectangular regions that can be integrated over.

To simplify, we only consider elementary regions of integration.

A region is called y-simple if you can describe it with 2 functions of x: $f_1(x)$, $f_2(x)$

where R is $(x, y) \in \mathbb{R}^2$ satisfying

$$f_1(x) \leq y \leq f_2(x) \quad \text{and} \quad x \in [a, b]$$



A region is x-simple if there are $g_1(y)$, $g_2(y)$ so that:

R is $(x, y) \in \mathbb{R}^2$ satisfying

$$g_1(y) \leq x \leq g_2(y)$$

A region is simple if it is x-simple and y-simple.

Integrating over some simple region

$y \in [c, d]$ $\quad g_1(y) \quad g_2(y)$

$$\iint \quad \int^{d} \int^{g_2(y)}$$

$$\iint_R f(x,y)\,\partial y\,\partial x = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)\,\partial y \right) \partial x$$

$x \in [c, d]$ $\quad f_1(x) \quad f_2(x)$

$$\iint_R f(x,y)\,\partial x\,\partial y = \int_c^d \left(\int_{f_1(x)}^{f_2(x)} f(x,y)\,\partial x \right) \partial y$$

If R is x-simple and y-simple:

$$\iint_R f(x,y)\,\partial y\,\partial x = \int_{g_1(x)}^{g_2(x)} \int_{f_1(x)}^{f_2(x)} f(x,y)\,\partial x\,\partial y$$

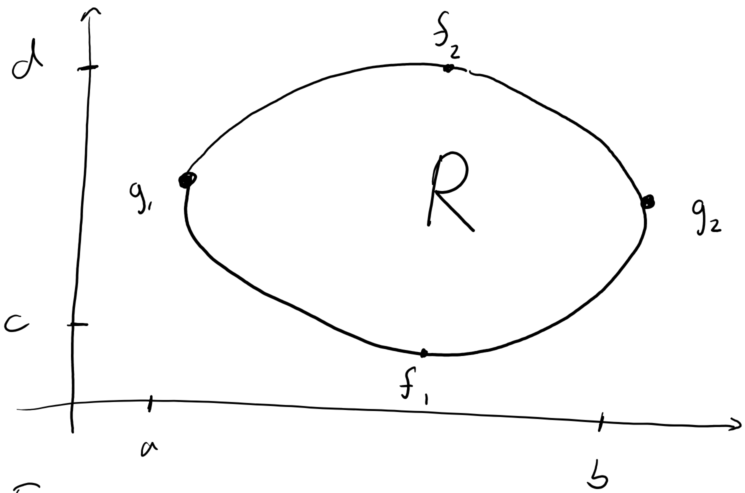
$$= \int_{f_1(x)}^{f_2(x)} \int_{g_1(x)}^{g_2(x)} f(x,y)\,\partial x\,\partial y$$

Changing the order of integration

Start with a simple region

- x simple: top and bottom are $f_1(x)$, $f_2(x)$

y simple: left and right are $g_1(x)$, $g_2(y)$



For any $f(x,y)$:

$$\iint_R f(x,y)\,dy\,dx : \int_a^b \int_{f_2}^{f_2} f(x,y)\,dy\,dx$$

$$= \int_c^d \int_{g_1}^{g_2} f(x,y)\,dx\,dy$$

Triple Integrals

$$R = [a, b] \times [c, d] \times [e, f]$$

so
$$a \le x \le b$$
$$c \le y \le d$$
$$e \le z \le f$$

Cut $[a, b] \times [c, d] \times [e, f]$ into n equal pieces (Δx, Δy, Δz apart)

$$\iiint_R f(x, y, z)\, dz\, dy\, dx$$

$$= \lim_{n \to \infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(x_i, y_i, z_i)\, \Delta z\, \Delta y\, \Delta x \right)$$

The triple integral can be integrated iteratively

$$\int_a^b \int_c^d \int_e^f f(x, y, z)\, dz\, dy\, dx$$

If R is a box, integration can be in any order