Linear Equations

Monday, January 4, 2021 9:20 AM

<u>Ex. Linear Equations</u>

$y = 2x - 1$

$3x - 5y + z = 4$

<u>Ex. Not Linear Equations</u>

$y = x^2$

$\tan(y) + xz = 3$

<u>Def.</u>

A linear equation is of the form

$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 \dots a_n x_n = b$

where $\underbrace{a_1, a_2, \dots a_n}_{\text{coefficients}}$ and b are constants

and $x_1, x_2 \dots x_n$ are variables

System of Linear Equations

$x - y = 3$ $x + y = 3$	$2x - 2y = 6$ $-2x + 2y = -6$	$x - y = 3$ $x - y = 2$
$(3, 0)$	$\mathbb{R}$	No Sol'ns
one solution	infinite solutions	no solutions

<u>Def.</u>

A system of linear equations is a collection of linear equations

$\left[a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \right]$ m linear equations

$$\left[\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right]$$

m linear equations in n unknowns

A <u>solution</u> to the linear system is an ordered n-tuple $(s_1, s_2 \dots s_n)$ such that substituting x_n for s_n then every equation is true

The <u>solution set</u> of the system is the set of all solutions of the system

Two systems are equivalent if they have the same solution set.

System Consistency

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Any linear system has:

- exactly one solution
- infinitely many solutions
- no solution

Def.

A linear system is consistent if it has at least one solution. Otherwise it is inconsistent.

Matrix

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Def. A matrix is a rectangular array of numbers.
An $m \times n$ matrix is a matrix with m rows and n columns.

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix.

Def. Given the linear system:

$$\begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \end{bmatrix}$$

its coefficient matrix is:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

its augmented matrix is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Solving Linear Systems

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1. Adding a multiple of one equation to another
2. Interchange two equations.
3. Multiply an equation by a non 0 constant

Elementary Row Operations

Monday, January 4, 2021 9:45 AM

1. Add a multiple of one row to another
2. Swap two rows
3. Multiply entries in a row by a non 0 constant

<u>Def.</u> two matrices are row equivalent if we can obtain one matrix from the other using elementary row operations.

Row Echelon Forms

Wednesday, January 6, 2021 9:46 PM

Def. A matrix is in echelon/row echelon form if:

1. All nonzero rows are above any zero row(s)
2. Each $\underbrace{\text{leading entry}}_{\text{the first nonzero entry } l \to r}$ of a nonzero row is in a column to the right of the leading entry above it.
3. All entries in a column below a leading entry are zeros

Examples:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

Def. A matrix is in reduced echelon/reduced row echelon form if it is in row echelon form

4. The leading entry of each nonzero row is 1
5. Each leading 1 is the only only nonzero entry in its column.

Examples:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian Elimination

Wednesday, January 6, 2021 10:02 PM

Gaussian elimination / row reduction is the process of transforming a matrix into row echelon form using elementary row operations

Def. Let A be a matrix and B be its rref. For any leading 1 in B, the corresponding location in A is a pivot position and the column where it is located is a pivot column.

A pivot is a nonzero entry in a pivot position that is used to create zeros

Steps: $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & -4 & 2 \\ 2 & -3 & 5 & 1 & 0 \\ 1 & 0 & 1 & 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & -4 & 2 \\ 2 & -3 & 5 & 1 & 0 \\ 1 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Step 1 Find the leftmost nonzero column. (This is a pivot column.)

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 4 & -4 & 2 \\ 2 & -3 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2 Choose a nonzero entry in this column to be a pivot. Interchange rows such that the pivot is at the top

$R_1 \longleftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & -8 & -4 \\ 0 & -3 & 3 & -3 & -6 \end{bmatrix}$$

Step 3 Use the pivot to creat zeros below it.

$-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 0 & 0 & 2 & -8 & -4 \\ 0 & -3 & 3 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array}$$

zeros below it.

$$\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{1} & \cancel{2} & \cancel{3} \\ 0 & 0 & 2 & -8 & -4 \\ 0 & -3 & 3 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 4 Ignore the row with the pivot position and any rows above it. Repeat steps 1-3 until the final row is reached

$$\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{0} & \cancel{-3} & \cancel{3} & \cancel{-3} & \cancel{-6} \\ 0 & 0 & 2 & -8 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

(fulfils step 3)

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & -3 & 3 & -3 & -6 \\ 0 & 0 & 2 & -8 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{ref}$$

The matrix is now in row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & -3 & 3 & -3 & -6 \\ 0 & 0 & 2 & -8 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5 To transform this into rref, start with the rightmost pivot, create zeros. Repeat from right to left. Rescale remaining rows so that all pivots are 1.

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & -3 & 3 & -3 & -6 \\ 0 & 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \tfrac{1}{2}R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 6 & 5 \\ 0 & -3 & 0 & 9 & 0 \\ 0 & 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} -3R_3 + R_2 \to R_2 \\ -R_3 + R_1 \to R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 6 & 5 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \tfrac{1}{3} R_2 \to R_2$$

The matrix is now in reduced row echelon form.

Solutions, Existence, Uniqueness

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Ex. $x - y = 3$
$-2x + 2y = -6$ $\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ -2 & 2 & -6 \end{array}\right]$ $2R_1 + R_2 \rightarrow R_1$

$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right]$ $x - y = 3$, $x = 3 + y$

If we assign any value to y, then x is determined.

solution set $= \{ (3+t, t) \mid t \in \mathbb{R} \}$

Def. Take any linear system and its augmented matrix. Variables corresponding to pivot columns are basic variable. Otherwise they're free variables.

Ex. In the linear system with x_1, x_2, x_3, x_4,

$$\left[\begin{array}{cccc|c} -1 & -4 & 1 & -12 & 4 \\ 1 & 4 & 1 & -2 & 2 \end{array}\right]$$

rref: $\left[\begin{array}{cccc|c} 1 & 4 & 0 & 5 & -1 \\ 0 & 0 & 1 & -7 & 3 \end{array}\right] \rightarrow$ $x_1 + 4x_2 + 5x_4 = -1$
$x_3 - 7x_4 = 3$

basic variables: x_1, x_3
free variables: x_2, x_4

$x_1 = -1 - 4x_2 - 5x_4$
$x_3 = 3 + 7x_4$

solution: $\begin{cases} x_1 = -1 - 4x_2 - 5x_4 \\ x_2 \text{ is free} \\ x_3 = 3 + 7x_4 \\ x_4 \text{ is free} \end{cases}$

Theorem (Existence and Uniqueness)

- A linear system is consistent iff the right most column of the augmented matrix is not a pivot column. ie. there is no row of the form $[0 \dots 0 \quad b]$ in its row echelon form.
- If a linear system is consistent, then either it has:
  1) Exactly one solution (unique solution), if there are no free variables

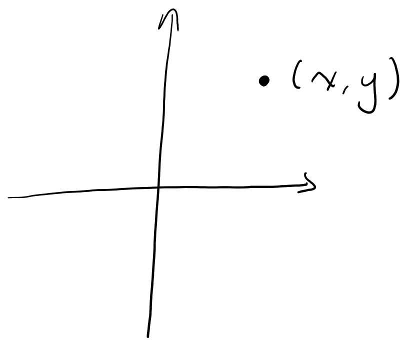
2) Infinitely many solutions, there is a free variable.

Vectors

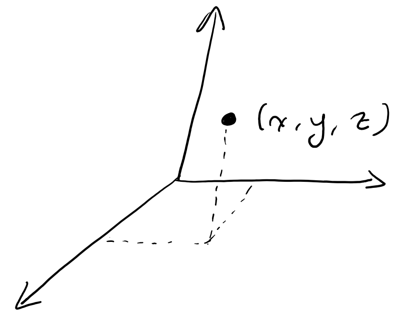
Saturday, January 9, 2021 4:13 PM

$\mathbb{R}$

$\mathbb{R}^2$



$\mathbb{R}^3$



Def. $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers.

$$\mathbb{R}^n = \{ (u_1, u_2, \dots, u_n) \mid u_i \in \mathbb{R} \text{ for } 1 \le i \le n \}$$

$\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are equal if $u_i = v_i$ for $1 \le i \le n$

We will call elements in $\mathbb{R}^n$ as vectors

Notation:

$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ Column Vector $(n \times 1)$ $\qquad$ $[u_1, u_2 \dots u_n]$ Row Vector $(1 \times n)$

"$x \in A$" means that x is an element in the set A

Vector Operations

Saturday, January 9, 2021 4:21 PM

Addition

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \left.\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}\right\} \in \mathbb{R}^n$$

Scalar Multiplication

$$c\vec{u} = c \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \left.\begin{bmatrix} c \cdot u_1 \\ c \cdot u_2 \\ \vdots \\ c \cdot u_n \end{bmatrix}\right\} \in \mathbb{R}^n$$

Subtraction

$$\vec{u} - \vec{v} = u + (-1 \cdot \vec{v}) = \left.\begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{bmatrix}\right\} \in \mathbb{R}^n$$

Property of Vectors

Saturday, January 9, 2021 4:25 PM

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and c, d be scalars

1) (Addition is commutative) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2) (Addition is associative) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3) (Additive identity exists) $\vec{u} + \vec{0} = \vec{u}$

4) (Additive inverse exists) $\vec{u} + (-\vec{u}) = \vec{0}$

5) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

6) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

7) $c(d\vec{u}) = (cd)\vec{u}$

8) $1\vec{u} = \vec{u}$

Linear Combination

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Def. $\vec{y} \in \mathbb{R}^n$ is a linear combination of $\vec{v_1}, \vec{v_2} \dots \vec{v_p} \in \mathbb{R}^n$ if there exists scalars $c_1, c_2, \dots c_p$ such that:

$$\vec{y} = c_1 \vec{v_1} + c_2 \vec{v_2} + \dots c_p \vec{v_p}$$

Span

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<u>Def</u>. The <u>span</u> of $\vec{v_1} \ldots \vec{v_p} \in \mathbb{R}^n$ is the set of all linear combinations of $\vec{v_1} \ldots \vec{v_p}$.

$$\text{span}\{\vec{v_1} \ldots \vec{v_p}\} = \{c_1\vec{v_1} + c_p\vec{v_p} \mid c_i \in \mathbb{R} \text{ for } 1 \leq i \leq p\}$$

The set is <u>spanned/generated</u> by $\vec{v_1} \ldots \vec{v_p}$

Ax = b

Wednesday, January 13, 2021 12:25 PM

Def. Let A be an $m \times n$ matrix and $\vec{a_1} \ldots \vec{a_n}$ be its columns. Let $\vec{x} \in \mathbb{R}^n$. Then the product of A and $\vec{x}$ is:

$$A\vec{x} = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x_1}\vec{a_1} + \vec{x_2}\vec{a_2} \ldots + \vec{x_n}\vec{a_n}$$

which is a linear combination of colums of A with weigts $\vec{x_1} \ldots \vec{x_n}$

Properties of $A\vec{x}$

1) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

2) $A(c\vec{u}) = c(A\vec{u})$

Def Given some vector equation $x_1\vec{a_1} + \ldots x_n\vec{a_n} = \vec{b}$ it can be rewritten as:

$$\begin{bmatrix} \vec{a_1} & \ldots & \vec{a_n} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

and has the same solution set as:

- Linear system
- Vector equation

Theorem The matrix equation $A\vec{x} = \vec{b}$ has a solution iff is a linear combination of the columns of A (i.e. $\vec{b}$ is in the span of the columns of A)

Theorem Let A be an $m \times n$ matrix then the following statements are equivalent (all statements are all true or false)

1) For any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution

2) Every $\vec{b}$ [illegible] $\mathbb{R}^m$ [illegible]

1) For any $b \in \mathbb{R}$, $A\vec{x} = b$ has a solution

2) Every $\vec{b}$ in $\mathbb{R}^n$ is a linear combination of cols A

3) Columns of A span $\mathbb{R}^m$ (ie $\text{Span}\{\vec{a_1}, \dots \vec{a_n}\} = \mathbb{R}^m$)

4) A has a pivot position in every row.

Identity Matrix

Wednesday, January 13, 2021 12:44 PM

Def. An identity matrix I_n is an $n \times n$ matrix with 1's on the diagonal starting from the upper left corner and 0's everywhere else.

$$I_n \vec{x} = \vec{x}$$

I_n acts like 1 in multiplication

Solutions of Homogenous Systems

Wednesday, January 13, 2021 12:47 PM

<u>Def.</u> A linear system is homogeneous if all constant terms are zero. $A\vec{x} = \vec{0}$

Notice: Any homogenous system is consistent.

<u>Def.</u> The trivial solution of $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

A nontrivial solution of $A\vec{x} = \vec{0}$ is any solution $\neq \vec{0}$

<u>Theorem</u> $A\vec{x} = \vec{0}$ has a nonzero solution iff the system has at least one free variable.

<u>Theorem</u> Suppose that $A\vec{x} = \vec{b}$ is consistent & let $\vec{p}$ be a solution. Then:

solution set of $A\vec{x} = \vec{b}$ $= \{\vec{p} + \vec{v}_n \mid \vec{V_n}$ is a solution of $A\vec{x} = \vec{0}\}$

Proof:

If $\vec{q}$ is a soln of $A\vec{x} = \vec{b}$, then
$$\vec{q} = \vec{q} + \vec{p} - \vec{p}$$
$$= \vec{p} = (\vec{q} - \vec{p})$$

$$A(\vec{q} - \vec{p}) = A\vec{q} - A\vec{p}$$
$$= \vec{b} - \vec{b} = \vec{0}$$

therefore $\vec{q} - \vec{p}$ is a soln of $A\vec{x} = \vec{0}$

If $\vec{q} = \vec{p} + \vec{V_n}$ for some soln. $\vec{V_n}$ of $A\vec{x} = \vec{0}$ then $\vec{q}$ is a soln. to $A\vec{x} = \vec{b}$ b/c

$$A\vec{q} = A(\vec{p} + \vec{V_n})$$
$$= A\vec{p} + A\vec{V_n}$$
$$= \vec{b} + \vec{0}$$
$$= \vec{b}$$

Linear Dependence/Independence

Sunday, January 17, 2021 4:58 PM

Consider the following subsets of $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ Notice that neither $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ nor $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a linear combination of the other

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ There is a vector which is a linear combination of the others

Def. $\{ \vec{v_1} \dots \vec{v_p} \} \subseteq \mathbb{R}^n$ is <u>linearly independent</u> if

$x_1 \vec{v_1} + x_2 \vec{v_2} + \dots x_p \vec{v_p} = \vec{0}$ has only the trivial solution

(i.e. $x_1, x_2, \dots x_p$ must be zero) otherwise, it is

<u>linearly dependent</u>

A <u>linear dependence</u> relation is an equation $c_1 \vec{v_1} + \dots c_p \vec{v_p} = \vec{0}$

where $c_1, \dots, c_p$ are not all zero.

Theorems

Sunday, January 17, 2021 5:24 PM

Set w/ Exactly One Vector

$\{\vec{v}\}$ is LI iff $\vec{v} \neq \vec{0}$

Set w/ Exactly Two Vectors

$\{\vec{v_1}, \vec{v_2}\}$ is LD iff at least one vector is a scalar multiple of the other

Set w/ Two or More Vectors

Theorem (Characterization of LD sets)

Let $S = \{\vec{v_1}, \dots, \vec{v_p}\}$ where $p \geq 2$ then S is LD iff at least one vector in S is a linear combination of the other vectors in S.

Theorem If $S = \{\vec{v_1}, \dots, \vec{v_p}\} \subseteq \mathbb{R}^n$ and $p > n$ then S is LD

Theorem If $S = \{\vec{v_1}, \dots, \vec{v_p}\}$ contains $\vec{0}$ then S is LD

Theorem The columns of a matrix A are LI iff $A\vec{x} = \vec{0}$ has only the trivial solution.

Matrix Transformation

Wednesday, January 20, 2021 7:24 PM

Def. A <u>transformation / function / mapping</u> from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each $\vec{x} \in \mathbb{R}^n$ exactly one vector $T(\vec{x}) \in \mathbb{R}^m$

Notation: $T: \mathbb{R}^n \to \mathbb{R}^m$

Domain ($\mathbb{R}^n$), Codomain ($\mathbb{R}^m$)

$\underset{\in \mathbb{R}^n}{\vec{x}} \longmapsto \underset{\in \mathbb{R}^m}{T(\vec{x})}$ ← image of $\vec{x}$ under T

$$\text{Range of } T = \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n \}$$

Def. A <u>matrix transformation</u> is a map $T: \mathbb{R}^n \to \mathbb{R}^m$ where $\vec{x} \longmapsto A\vec{x}$ for some matrix A. That is, $T(\vec{x}) = A\vec{x}$.

Notice: Given an $m \times n$ matrix A, if $T: \mathbb{R}^n \to \mathbb{R}^m$ is the map $T(\vec{x}) = A\vec{x}$, then

$$\begin{aligned} \text{Range of } T &= \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n \} \\ &= \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \} \\ &= \text{Span of columns of } A \end{aligned}$$

Linear Transformations

Wednesday, January 20, 2021 7:42 PM

Def. A transformation $T: V \to W$ is linear if

1) $T(\vec{u}+\vec{v}) = T(u) + T(v)$ / $\vec{u}, \vec{v} \in V$
2) $T(c\vec{u}) = cT(\vec{u})$ / $\vec{u} \in V$, any scalar c

Theorem For any linear transformation, $T: V \to W$

1) $T(\vec{0}) = \vec{0}$
2) $T(c_1\vec{v_1} + \dots + c_p\vec{v_p}) = c_1T(\vec{v_1}) + \dots c_pT(\vec{v_p})$

for any vectors $\vec{v_1}, \dots, \vec{v_p} \in V$ & scalars $c_1, \dots, c_p$

Standard Matrix

Wednesday, January 20, 2021 7:59 PM

Notation: In $\mathbb{R}^n$, $\vec{e_j}$ is the vector whose jth entry is one with 0's everywhere else. Also are columns of the identity matrix.

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

<u>Theorem</u> Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that:

$T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^n$

In fact, A is the $m \times n$ matrix whose jth column is $T(\vec{e_j})$:

$$A = \left[\, T(\vec{e_1}) \,\middle|\, T(\vec{e_2}) \,\middle|\, \dots \,\middle|\, T(\vec{e_n}) \,\right]$$

This is called the standard matrix for T

Linear transformations are completely determined on how they act on the standard basis vectors $(\vec{e_1}, \dots \vec{e_n})$

Onto and One-to-One

Wednesday, January 20, 2021 8:09 PM

Def. $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if every $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x} \in \mathbb{R}^n$

$T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one iff $T(\vec{x}) = \vec{0}$ has only the trivial soln.

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation w/ std matrix A. Then

1) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$

2) T is one-to-one iff the columns of A are linearly independent

Special Matrices and Equality

Sunday, January 24, 2021 6:55 PM

$$A_{nm} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} = entry in the ith row and jth column of A

$a_{11}, a_{22}, a_{33} \ldots$ are the diagonal entries of A

<u>Def</u> A <u>diagonal matrix</u> is a square matrix whose non diagonal entries are zero.

A <u>zero matrix</u> (denoted by O), is a matrix whose entries are all zeroes.

Two $m \times n$ matrices A & B are <u>equal</u> if

$$(A)_{ij} = (B)_{ij} \quad \text{for all} \quad 1 \le i \le m \;\&\; 1 \le j \le n$$

Matrix Operations

Sunday, January 24, 2021 7:00 PM

Let A & B be $m \times n$ matrices & c be a scalar, then:

- $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$
- $(cA)_{ij} = c(A)_{ij}$

<u>Properties</u>

commutative: $A+B = B+A$

associative: $(A+B)+C = A+(B+C)$

zero : $A+0 = A$

distributive: $r(A+B) = rA + rB$

distributive: $(r+s)A = rA + sA$

distributive: $r(sA) = (rs)A$

Matrix Multiplication

Sunday, January 24, 2021 7:13 PM

Reasoning

$T(\vec{x}) = A\vec{x}$

$U(\vec{x}) = B\vec{x}$

The composition $\underbrace{T \circ U : \mathbb{R}^p \to \mathbb{R}^m}_{\mathbb{R}^p \xrightarrow[U]{} \mathbb{R}^n \xrightarrow[T]{} \mathbb{R}^m}$ is defined by

$(T \circ U)(\vec{x})$ or $(TU)(\vec{x}) = T(U(\vec{x})) = T(B(\vec{x})) = A(B\vec{x})$

$= A(x_1 \vec{b_1} + x_2 \vec{b_2} \cdots + x_p \vec{b_p})$

$= A(x_1 \vec{b_1}) + \ldots\ldots A(x_p \vec{b_p})$

$= x_1(A\vec{b_1}) + \ldots\ldots x_p(A \vec{b_1})$

$= \underbrace{\left[A\vec{b_1} \mid \ldots\ldots \mid A\vec{b_p} \right]}\cdot \vec{x}$

$= A \circ B$ or AB

Def Let A be an $m \times n$ matrix & B be an $n \times p$ matrix.

Then $AB = A[\vec{b_1} \mid \ldots\ldots \mid \vec{b_p}] = [A\vec{b_1} \mid \ldots\ldots \mid A\vec{b_p}]$

which is an $m \times p$ matrix

Row-Column Rule for AB

Let A be an $m \times n$ matrix & B be an $n \times p$ matrix.

Then $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} \times b_{kj}$

which is the dot product of the ith row of A with jth column of B

Properties of Matrix Multiplication

Sunday, January 24, 2021 7:20 PM

Let A be an $n \times n$ matrix & B, C be matrices so that the following sums & products are defined:

1) $A(BC) = (AB)C$
2) $A(B+C) = AB + AC$
3) $(B+C)A = BA + CA$
4) For any scalar r, $r(AB) = (rA)B = A(rB)$
5) $I_m A = A = I_n$

However, in general:

1) $AB \neq BA$
2) $AB = AC \not\Rightarrow B = C$
3) $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

Transpose

Sunday, January 24, 2021 7:27 PM

Def Let A be a square matrix & K be a positive integer. Then

$$A^K = \underbrace{A \cdots A}_{K \text{ copies}}$$

Def Let A be an $m \times n$ matrix. The transpose of A (denoted A^T) is the $n \times m$ matrix whose ith column is the ith row of A. That is,

$$(A^T)_{ij} = A_{ji}$$

Properties of transpose

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) For any scalar r, $(rA)^T = r(A^T)$
4) $(AB)^T = B^T A^T$

Inverse

Wednesday, January 27, 2021 9:49 PM

<u>Def.</u> An $n \times n$ matrix A is <u>invertible/nonsingular</u> if there exists an $n \times n$ matrix such that $AC = I$ & $CA = I$

If such a C exists, then it is unique & we call it the <u>inverse</u> of A, denoted A^{-1}

If A is not invertible, we say that A is singular

<u>Theorem</u> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $\det(A) = ad - bc$. Then

A is invertible iff $\det(A) \neq 0$

If A is invertible, then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Properties of the Inverse

Wednesday, January 27, 2021 9:57 PM

<u>Theorem</u>. If A is an invertible $n \times n$ matrix, then for each $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has the unique soln. $\vec{x} = A^{-1}\vec{b}$

<u>Theorem</u>. Let A & B be invertible $n \times n$ matrices. Then

1) A^{-1} is invertible & $(A^{-1})^{-1} = A$

2) AB is invertible & $(AB)^{-1} = B^{-1}A^{-1}$

3) A^T is invertible & $(A^T)^{-1} = (A^{-1})^T$

<u>Use</u>: $A \cdot A^{-1} = I$ & $A^{-1} \cdot A = I$ where the I are the same

Invertibility and Elementary Matrices

Wednesday, January 27, 2021 10:02 PM

Theorem If $A_1, A_2, \dots, A_k$ are invertible $n \times n$ matrices, then $A_1 A_2 \dots A_k$ is invertible & $(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1} A_1^{-1}$

Def An $n \times n$ elementary matrix is obtained by performing a single elementary row operation to I_n

Ex.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-2R_1 + R_2 \to R_2 \qquad R_1 \leftrightarrow R_2 \qquad 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

When you multiply a matrix by an elementary matrix;

EA = the matrix obtained by performing the same row operation done on E to A

Inverse as Elementary Matrices

Wednesday, January 27, 2021 10:09 PM

E^{-1} = elementary matrix corresponding to the elementary row operation that will reproduce I (opposite of the first row operation).

Theorem $A_{n\times n}$ is invertible iff A is row equivalent to I_n
aka A can be row reduced to I_n.

Notice: $A^{-1} = (E_1^{-1} \cdots E_p^{-1})^{-1} = E_p \cdots E_1 = E_p \cdots E_1 \cdot I$

Calculating Inverse

Wednesday, January 27, 2021 10:16 PM

How to find A^{-1}

If possible, row reduce $[A \mid I]$ to $[I \mid A^{-1}]$

Invertible Matrix Theorem

Wednesday, January 27, 2021 10:21 PM

Theorem. (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. Then the following statements are equivalent

1) A is invertible
2) $A \sim I_n$ (row equivalent)
3) A has n pivot positions
4) $A\vec{x} = \vec{0}$ has only trivial soln
5) Columns of A are LI
6) $\vec{x} \mapsto A\vec{x}$ is one-to-one
7) $A\vec{x} = \vec{b}$ has a solution for $\vec{b} \in \mathbb{R}^n$
8) Columns of A span $\mathbb{R}^n$
9) $\vec{x} \mapsto A\vec{x}$ is onto
10) There's some C such that $CA = I_n$
11) There's $D_{n \times n}$ such that $AD = I_n$
12) A^T is invertible

Invertible Linear Transformations

Wednesday, January 27, 2021 10:27 PM

Def. The map $T:\mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists $S:\mathbb{R}^n \to \mathbb{R}^n$ such that $T(S(\vec{x})) = \vec{x}$ & $S(T(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$

In this case, S is unique & we call it the inverse of T, denoted T^{-1}

Theorem Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation w/ standard matrix A, then T is invertible iff A is invertible.

- In this case, then T^{-1} is a linear transformation & its standard matrix is A^{-1}

Vector Space Properties

Tuesday, February 2, 2021 6:29 PM

Def. A <u>vector space</u> is a nonempty set V of objects, called <u>vectors</u>, w/ a set of scalars (e.g. $\mathbb{R}$), & operations called addition & scalar multiplication satisfy the following:
For any $\vec{u}, \vec{v}, \vec{w} \in V$ & scalars c, d

1) $\vec{u} + \vec{v} \in V$: closed under addition
2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$: addition is commutative
3) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$: addition is associative
4) There is some $\vec{0} \in V$ such that
$\vec{u} + \vec{0} = \vec{u}$: additive identity exists in V
5) For each $\vec{u} \in V$ there is $-\vec{u} \in V$ such that
$\vec{u} + -\vec{u} = \vec{0}$: additive inverse exists in V
6) $c\vec{u} \in V$: closed under scalar multiplication
7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
9) $c(d\vec{u}) = (cd)\vec{u}$
10) $1 \cdot \vec{u} = \vec{u}$

Zero Vector and Inverse Properties

Tuesday, February 2, 2021 6:38 PM

__Theorem__. Let V be a vector space. For any $\vec{u} \in V$ & scalar c:

a) $0\vec{u} = \vec{0}$ b) $c\vec{0} = \vec{0}$ c) $-\vec{u} = (-1)\vec{u}$

Examples of Real Vector Spaces

Tuesday, February 2, 2021 6:39 PM

Some Examples of (Real) Vector Spaces

Set of Vectors	Addition	Scalar Multiplication	Zero Vector
$\mathbb{R}^n = \{(x_1,\dots,x_n) \mid x_i \in \mathbb{R}$ for all $i=1,\dots,n\}$	$(x_1,\dots,x_n)+(y_1,\dots,y_n) = (x_1+y_1,\dots,x_n+y_n)$	$c(x_1,\dots,x_n)=(cx_1,\dots,cx_n)$	$(0,0,\dots,0)$
For $n \geq 0$, $\mathbb{P}_n$ is the set of all polynomials in t w/ real coefficients & degree at most n	$(a_0+a_1t+\dots+a_nt^n) + (b_0+b_1t+\dots+b_nt^n) = (a_0+b_0)+(a_1+b_1)t + \dots + (a_n+b_n)t^n$	$c(a_0+a_1t+\dots+a_nt^n) = ca_0+ca_1t+\dots+ca_nt^n$	$p(t)=0$, the zero polynomial
For any $\mathbb{D} \subseteq \mathbb{R}$, the set of all functions $f: \mathbb{D} \to \mathbb{R}$	$(f+g)(t) = f(t)+g(t)$	$(cf)(t) = cf(t)$	$f(t)=0$, the constant function zero
$\mathbb{P}$ is the set of all polynomials in t w/ real coefficients	usual addition of polynomials	usual scalar multiplication of polynomials	zero polynomial
The set of all doubly infinite sequences of real numbers $\{x_n\} = (\dots, x_{-1}, x_0, x_1, \dots)$	$\{x_n\}+\{y_n\} = \{x_n+y_n\}$	$c\{x_n\} = \{cx_n\}$	$(\dots,0,0,0,\dots)$
$M_{m\times n} = \{A \mid A$ is an $m\times n$ matrix w/ real	usual matrix addition	usual matrix scalar multiplication	$m\times n$ zero matrix

Vector Subspaces

Tuesday, February 2, 2021 6:51 PM

Def. A subspace of a vector space V is a subset H of V such that:

a) $\vec{0} \in H$

b) If $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$

c) If $\vec{u} \in H$, c scalar, $c\vec{u} \in H$

* A subspace of V is a subset which is a vector space

Examples

- Any vector space V has the zero subspace, $\{\vec{0}\}$
  - a) $\vec{0} \in \{\vec{0}\}$ is true
  - b) $\vec{0} + \vec{0} = \vec{0} \in \{\vec{0}\}$ is true
  - c) $c \cdot \vec{0} = \vec{0} \in \{\vec{0}\}$ is true
- V is a subspace of itself

Subspace Spanned by a Set

Tuesday, February 2, 2021 7:51 PM

Theorem. Let V be a vector space. If $\vec{v_1} \dots \vec{v_p}$, then $\text{Span}\{\vec{v_1} \dots \vec{v_p}\}$ is a subspace of V.

Def. The subspace generated/spanned by $\{\vec{v_1} \dots \vec{v_p}\}$ is $\text{Span}\{\vec{v_1} \dots \vec{v_p}\}$.

If H is a subspace of V & $H = \text{Span}\{\vec{v_1} \dots \vec{v_p}\}$, then $\{v_1 \dots v_p\}$ is a generating/spanning for H.

Null, Column, Row Spaces

Tuesday, February 2, 2021 7:57 PM

Def. Let A be an $m \times n$ matrix. The null space of A is

$$\text{Nul } A = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$$

Theorem. If A is an $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^n$

Def. Let A be an $m \times n$ matrix. The column space of A is

$$\text{Col } A = \text{Span}\{ \vec{a_1} \dots \vec{a_n} \} = \{ \vec{b} \in \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ for } \vec{x} \in \mathbb{R}^n \}$$

Theorem. If A is an $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^m$.

Def. Let A be an $m \times n$ matrix. The row space of A is

$$\text{Row } A = \text{Span}\{ \vec{r_1} \dots \vec{r_m} \}$$

Theorem. If A is an $m \times n$ matrix, then Row A is a subspace of $\mathbb{R}^n$

Kernel, Range of Linear Transformation

Tuesday, February 2, 2021 8:14 PM

Recall: A linear transformation is a map $T: V \to W$ where V & W are vector spaces such that

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = cT(\vec{u})$$

Def. The kernel/nullspace of $T: V \to W$ is the set

$$\{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \}$$

The range of T is the set $\{ T(\vec{x}) \mid \vec{x} \in V \}$ or equivalently,

$$\{ \vec{y} \in W \mid T(\vec{x}) = \vec{y} \quad \vec{x} \in V \}$$

kernel/nullspace : $Nul(A)$ range: $Col(A)$

Theorem If $T: V \to W$ is a linear transformation, then the kernel of T is a subspace of V & the range is also a subspace of W.

Linear Independence/Dependence

Thursday, February 4, 2021 10:09 PM

Def A subset $\{\vec{v_1} \dots \vec{v_p}\}$ of a vector space V is <u>linearly independent</u> if $c_1\vec{v_1} + \dots + c_p\vec{v_p} = \vec{0}$ has only the trivial solution. Otherwise it is linearly dependent & any equation $c_1\vec{v_1} + \dots + c_p\vec{v_p} = \vec{0}$ w/ at least one non trivial solution is a linear dependence relationship.

Facts

- $\{\vec{v}\}$ is LI iff $\vec{v} \neq \vec{0}$
- $\{\vec{u}, \vec{v}\}$ is LI iff neither vector is a scalar multiple of the other
- If $\vec{0} \in S$, the S is LD
- If $S = \{\vec{v_1} \dots \vec{v_p}\}$ has at least two vectors, S is LD iff at least one vector in S is a linear combination of the other vectors in S.

Basis

Thursday, February 4, 2021 10:17 PM

Some standard basis:

1) $V = \mathbb{R}^n$: $\{ \vec{e_1}, \vec{e_2}, \dots, \vec{e_n} \}$

2) $V = P_n$: $\{ 1, t, t^2 \dots t^n \}$

3) $V = M_{2\times 3}$: $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

<u>Def.</u> Let H be a subspace of a vector space V. Then $\beta = \{ \vec{b_1} \dots \vec{b_p} \} \subseteq V$ is a basis for H if

1) β is LI
2) Span $\{ \vec{b_1} \dots \vec{b_p} \} = H$

How to find Basis For A

Thursday, February 4, 2021 10:53 PM

Given a matrix A.

Theorem The basis for Nul A is the solution for $A\vec{x} = 0$ in the general solution or parametric vector form.

Theorem The pivot columns of A form a basis for Col A

Theorem If $A \sim B$ is in row echlon form, then the nonzero rows of B form a basis for Row A & Row B.

More general application: If $H = \text{Span}\{\vec{a_1} \ldots \vec{a_p}\}$ where $\vec{a_1} \ldots \vec{a_p} \in \mathbb{R}^n$, then matrix $A = [a_1 | \ldots | a_p]$.

The pivot columns will form a basis for H

Spanning Set Theorem

Thursday, February 4, 2021 10:37 PM

Spanning Set Theorem

Let $S = \{\vec{v_1} \dots \vec{v_p}\} \subseteq V$ & $H = \text{Span}\{\vec{v_1} \dots \vec{v_p}\}$

a) If some vector $\vec{v_k} \in S$ is a linear combination of the other vectors in S, then $\{\vec{v_1} \dots \vec{v_{k-1}}, \vec{v_{k+1}} \dots \vec{v_p}\}$ still spans H

b) If $H \neq \{\vec{0}\}$, then some subset of S spans H

Theorem

If $A \sim B$, then $A\vec{x} = \vec{b}$ & $B\vec{x} = \vec{b}$ have the same solutions

In particular, $x_1\vec{a_1} + \dots + x_n\vec{a_n} = \vec{0}$ iff $x_1\vec{b_1} + \dots x_n\vec{b_n} = \vec{0}$

Theorem

If $A \sim B$, then $\text{Row } A = \text{Row } B$

Dimensions of a Vector Space

Tuesday, February 9, 2021 2:19 PM

Can a vector space have more than one basis? Yes.

Is the number of vectors in a basis unique? Yes.

Theorem. If V is a vector space w/ basis $\beta = \{\vec{b}_1 \ldots \vec{b}_n\}$ then any set in β containing more than n vectors is linearly dependent.

Theorem. If V is a vector space w/ basis of size n, then every basis of V has exactly n vectors.

Def. A vector space is finite-dimensional if it is spanned by a finite set. Otherwise it's infinite-dimensional.

If V is finite dimensional, then the dimension of V is the number of vectors in a basis for V.

The dimension of $\{\vec{0}\}$ is 0.

Notation: $\dim(\mathbb{R}^n) = n$

$\dim(\mathbb{P}_n) = n+1$

$\dim(M_{m\times n}) = m\cdot n$

$\mathbb{P}$ is infinite-dimensional

Subspaces of Finite Dimensional Space

Tuesday, February 9, 2021 2:32 PM

<u>Theorem</u> Let H be a subspace of a finite-dim vector space V.
Then:
1) Every LI subset of H can be enlarged to a basis for H
2) H is also finite dimensional & $\dim(H) \le \dim(V)$

<u>Example</u> Subspaces of $\mathbb{R}^2$:
dim 0: $\{\vec{0}\}$ (origin)
dim 1: Span$\{\vec{v}\}$ where $v \neq 0$ (line through origin)
dim 2: $\mathbb{R}^2$ (plane)

<u>Theorem</u> Let V be a p-dim. vector space where $p \ge 1$. Then:
1) If $S \subseteq V$ has p elements & S is LI,
then S is a basis for V
2) If $S \subseteq V$ has p elements & S spans V,
then S is a basis for V

Rank and Nullity

Tuesday, February 9, 2021 2:38 PM

Def Let A be a matrix. The <u>rank</u> of A is $\dim(\text{Col } A)$ & the <u>nullity</u> of A is $\dim(\text{Nul } A)$

Theorem rank A = # of pivot columns/positions of A

nullity A = # of free variables in $A\vec{x} = \vec{0}$

Theorem Let A be an $m \times n$ matrix. Then

1) rank $A = \dim(\text{Row } A) = \text{rank}(A^T)$

2) (<u>Rank-Nullity Theorem</u>) rank A + nullity $A = n$

Invertible Matrix Theorem Cont.

Tuesday, February 9, 2021 2:52 PM

If A is an $n \times n$ matrix:

13) Columns of A form a basis for $\mathbb{R}^n$

14) $\text{Col } A = \mathbb{R}^n$

15) $\text{rank } A = n$

16) $\text{nul } A = \{\vec{0}\}$

17) $\text{nullity } A = 0$

Coordinate Systems

Tuesday, February 9, 2021 2:55 PM

<u>Theorem</u> (Unique representation theorem)

Let $\beta = \{\vec{b_1}, \ldots \vec{b_n}\}$ be a basis for a vector space V. Then for each $\vec{x} \in V$, there <u>exists</u> <u>unique</u> scalars $c_1 \ldots c_n$ such that $c_1 \cdot \vec{b_1} + \ldots + c_n \vec{b_n} = \vec{x}$

<u>Def</u> Let $\beta = \{\vec{b_1} \ldots \vec{b_n}\}$ be a basis for a vector space V.
Let $\vec{x} \in V$. If $\vec{x} = c_1\vec{b_1} + \ldots c_n\vec{b_n}$, then:

1) $c_1 \ldots c_n$ are the <u>coordinates</u> of $\vec{x}$ relative to basis β
2) $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the <u>coordinate vector</u> of $\vec{x}$ relative to β
3) $\vec{x} \longmapsto [\vec{x}]_\beta$ is the <u>coordinate mapping</u> determined by β

Change of Coordinate Matrix

Tuesday, February 9, 2021 3:10 PM

Def Let $\beta = \{\vec{b}_1 \dots \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. The change of coordinates matrix from β to the std. basis of $\mathbb{R}^n$ is:

$$P_\beta = [\vec{b}_1 | \dots | \vec{b}_n] \text{ \& } \vec{x} = P_\beta [\vec{x}]_\beta \text{ for any } \vec{x} \in \mathbb{R}^n$$

Theorem If P_β is the change of coord matrix from β to the standard matrix for $\mathbb{R}^n$, then $[\vec{x}]_\beta = P_\beta^{-1} \vec{x}$

Isomorphism

Tuesday, February 9, 2021 3:17 PM

Theorem Let $\beta = \{\vec{b_1} \dots \vec{b_n}\}$ be a basis for vector space V. Then the map $\vec{x} \longmapsto [\vec{x}]_\beta$ from V into $\mathbb{R}^n$ is a one-to-one and onto linear transformation

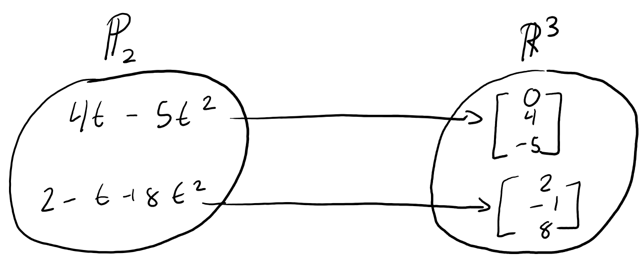
Def Let V & W be vector spaces. An isomorphism from V to W is a one-to-one and onto linear transformation $V \mapsto W$

We say that V is isomorphic to W if there exists an isomorphism from V onto W

Theorem Let V & W be finite-dim vector spaces. Then V is isomorphic to W iff $\dim V = \dim W$.

Example An isomorphism from $\mathbb{P}_2$ onto $\mathbb{R}^3$:

Take the coordinate mapping $T: \mathbb{P}_2 \to \mathbb{R}^3$ where $\beta = \{1, t, t^2\}$ & $T(p(t)) = [p(t)]_\beta$ is an isomorphism by the theorem above. $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$



In general

$$a_0 + a_1 t + a_2 t^2 \longmapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Change of Basis

Wednesday, February 10, 2021 9:08 PM

Given two different bases for a vector space, how are the coordinate vectors relative to one basis related to the coordinate vectors relative to the other?

<u>Theorem</u> Let $\beta = \{\vec{b}_1 \dots \vec{b}_n\}$, $\gamma = \{\vec{c}_1 \dots \vec{c}_n\}$ be bases for V. Then there exists a unique $n \times n$ matrix P such that $\underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta = [\vec{x}]_\gamma$ for $x \in V$.

$$\underset{\gamma \leftarrow \beta}{P} = \left[[\vec{b}_1]_\gamma \mid [\vec{b}_2]_\gamma \mid \dots \mid [\vec{b}_n]_\gamma \right]$$

which is called the <u>change-of-coordinates matrix</u> from β to γ

<u>Theorem</u> $\left(\underset{\gamma \leftarrow \beta}{P} \right)^{-1} = \underset{\beta \leftarrow \gamma}{P}$

Finding the Change of Basis Matrix

Wednesday, February 10, 2021 9:24 PM

Given $\beta = \{\vec{b}_1 \ldots \vec{b}_n\}$ and $\gamma = \{\vec{c}_1 \ldots \vec{c}_n\}$ to find ${}_{\gamma \leftarrow \beta}P$:

$$\begin{array}{c} \gamma \quad \leftarrow \quad \beta \\ \left[\vec{c}_1 \mid \ldots \mid \vec{c}_n \;\middle|\; \vec{b}_1 \mid \ldots \mid \vec{b}_n \right] \sim \left[I \;\middle|\; \underset{\gamma \leftarrow \beta}{P} \right] \end{array}$$

Recall: The change-of-coordinates matrix from $\beta = \{\vec{b}_1 \ldots \vec{b}_n\}$ to the standard basis for $\mathbb{R}^n$ is

$P_\beta = \left[\vec{b}_1 \mid \ldots \mid \vec{b}_n \right]$. For $\vec{x} \in \mathbb{R}^n$, $\vec{x} = P_\beta [\vec{x}]_\beta$

Using the more general notation, this is the same as

$$\underset{e_n \leftarrow \beta}{P} = \left[[\vec{b}_1]_{e_1} \mid \ldots \mid [\vec{b}_n]_{e_n} \right]$$

$$\underset{e_n \leftarrow \beta}{P} [\vec{x}]_\beta = [\vec{x}]_{e_n}$$

Example

Wednesday, February 10, 2021 9:13 PM

Take the bases $\beta = \left\{ \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\}$, $\gamma = \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

$$\underset{\gamma \leftarrow \beta}{P} = \left[[\vec{b}_1]_\gamma \mid [\vec{b}_2]_\gamma \right]$$

where $[\vec{b}_1]_\gamma$ and $[\vec{b}_2]_\gamma$ are the solutions to the linear system:

$$[\vec{\gamma}_1 \mid \vec{\gamma}_2]x = \vec{b}_1$$

and

$$[\vec{\gamma}_1 \mid \vec{\gamma}_2]x = \vec{b}_2$$

which are:

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ -5 & 2 & 5 \end{array}\right] \text{ and } \left[\begin{array}{cc|c} 1 & -2 & -3 \\ -5 & 2 & -1 \end{array}\right]$$

which can be solved simultaneously as:

$$\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$\to [\vec{b}_1]_\gamma$ $\quad \to [\vec{b}_2]_\gamma$

therefore

$$\underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} \quad \text{and} \quad \underset{\beta \leftarrow \gamma}{P} = \left(\underset{\gamma \leftarrow \beta}{P}\right)^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

Determinants

Sunday, February 14, 2021 9:35 PM

Recall: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

__Def__. Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the determinant of A, denoted by $\det A$ or $|A|$ as follows:

- If $n = 1$, then $\det A = \det[a_{11}] = a_{11}$
- If $n \geq 1$, then $\det A = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + \dots$

$$= \sum_{j=1}^{n} (-1)^{(j+1)} a_{ij} \cdot \det A_{ij}$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the 1st row and jth column of A

Cofactor Expansion

Sunday, February 14, 2021 9:42 PM

<u>Def</u>. The (i, j)-cofactor of A is $C_{ij} = (-1)^{(i+j)} \cdot \det A_{ij}$

<u>Theorem</u>. Let A be an $n \times n$ matrix where $n \geq 2$. Then

1) $\det A = \sum_{j=1}^{n} a_{ij} \cdot C_{ij}$ which is the <u>cofactor expansion across the ith row</u>

2) $\det A = \sum_{i=1}^{n} a_{ij} \cdot C_{ij}$ which is the <u>cofactor expansion across the jth column</u>

Determinant from Triangular Matrix

Sunday, February 14, 2021 9:51 PM

<u>Def.</u> A square matrix is triangular if either

1) All entries below the main diagonal are zeros (<u>upper triangular</u>)
2) All entries above the main diagonal are zeros (<u>lower triangular</u>)

<u>Theorem</u> If A is triangular, then $\det A$ is the product of the diagonal entries of A

Properties of Determinants

Wednesday, February 17, 2021 3:25 PM

<u>Theorem.</u> Let A be an $n \times n$ matrix.

a) If B is obtained by adding a multiple of one row of A to another row of A then $\det B = \det A$

b) If B is obtained by interchanging two rows of A then $\det B = -\det A$

c) If B is obtained by multiplying one row by scalar k then $\det B = k \cdot \det A$

<u>Theorem</u> If U is obtained from A using only row interchanges & replacements, and U is in row echelon form, then U is triangular &

$$\det A = \begin{cases} (-1)^r \cdot \det U & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases}$$

<u>Theorem</u> A is invertible iff $\det A \neq 0$

<u>Theorem</u> $\det(A^T) = \det A$

<u>Theorem</u> $\det(AB) = (\det A)(\det B)$

Linearity Properties

Wednesday, February 17, 2021 3:39 PM

Is the $\det : M_{n \times n} \mapsto \mathbb{R}$ a linear transformation? No

However, we get a linear transformation if we fix all but one column of a matrix: Let A be an $n \times n$ matrix.

Define $T : \mathbb{R}^n \mapsto \mathbb{R}$ by $T(\vec{x}) = \det [\vec{a_1} \mid \vec{a_2} \ldots \mid a_{j-1} \mid \vec{x} \mid a_{j+1} \ldots \mid a_n]$

This is a linear function b/c

- $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (by cofactor expansion)
- $T(c\vec{x}) = c\,T(\vec{x})$

General Formulas with Determinant

Wednesday, February 17, 2021 3:46 PM

<u>Some general formulas</u> if A is an $n \times n$ matrix.

- $\det(A^m) = (\det A)^m$
- $\det(kA) = k^n \cdot \det A$
- $\det(A^{-1}) = \dfrac{1}{\det A}$

Cramer's Rule and Inverse Formula

Wednesday, February 17, 2021 3:49 PM

<u>Cramer's Rule</u>: Let A be an $n\times n$ invertible matrix. Then for any $\vec{b}\in\mathbb{R}^n$, the unique solution $\vec{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ of $A\vec{x}=\vec{b}$ is given by $x_i = \dfrac{\det(A_i(\vec{b}))}{\det(A)}$ for $i=1,\dots,n$, where $A_i(\vec{b})$ is the matrix obtained by replacing the ith column of A w/ $\vec{b}$

<u>Inverse Formula</u>: Let A be an $n\times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\cdot \operatorname{adj} A$$

where the <u>adjugate/classical adjoint</u> of A is defined by:

$$(\operatorname{adj} A)_{ij} = C_{ji} = (C_{ij})^T$$

so $\operatorname{adj} A$ is the transpose of the matrix of cofactors.

Volume/Area of Parallelopiped Shapes

Wednesday, February 17, 2021 4:10 PM

Theorem Let $\vec{a_j}$ be the jth column of A

1) If A is a 2×2 matrix, then the area of the parallelogram determined by $\vec{a_1}$ & $\vec{a_2}$ is $|\det A|$

2) If A is a 3×3 matrix then the volume of the parallelpiped determined by $\vec{a_1}, \vec{a_2}, \vec{a_3}$ is $|\det A|$

Theorem For $T: V \mapsto W$ & $S \subseteq V$, let $T(S) = \{T(\vec{x}) \mid \vec{x} \in S\}$

1) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation w/ std matrix A. If S is a region in $\mathbb{R}^2$ w/ finite area, then

area of $T(S) = |\det A| \cdot$ area of S.

2) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation w/ std matrix A. If S is a region in $\mathbb{R}^3$ w/ finite volume, then

volume of $T(\vec{x}) = |\det A| \cdot$ volume of S.

Example Find the volume of the region bounded by the ellipsoid:

$$\frac{x_1^2}{2^2} + \frac{x_2^2}{3^2} + \frac{x_3^2}{4^2} = 1$$

Let S be the region bounded by $x_1^2 + x_2^2 + x_3^2 = 1$ ie. the unit ball

$T: \mathbb{R}^3 \mapsto \mathbb{R}^3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \vec{x}$, which means $T(\vec{x}) = S \mid \vec{x} \in E$

because $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_2 \\ 4x_3 \end{bmatrix} = \frac{(2x_1)^2}{2^2} + \frac{(3x_2)^2}{3^2} + \frac{(4x_3)^2}{4^2} = x_1^2 + x_2^2 + x_3^2$

so the volume of $E = |\det A| \cdot \text{vol } S = |2\cdot 3\cdot 4| \cdot \frac{4}{3}\pi = 32\pi$

Eigenvalue, Eigenvector, Eigenspace

Tuesday, February 23, 2021 8:16 PM

<u>Def</u>. Let A be an $n \times n$ matrix. An <u>eigenvector</u> of A is a non zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda \vec{x}$ for some scalar λ. We call λ an <u>eigenvalue</u> of A & we say that $\vec{x}$ is an <u>eigenvector corresponding to λ</u>

<u>Notice</u> $A\vec{x} = \vec{x}$ iff $A\vec{x} - \lambda\vec{x} = \vec{0}$ iff $A\vec{x} - \lambda I \vec{x} = \vec{0}$ iff $(A - \lambda I)\vec{x} = \vec{0}$

This implies that:

- λ is an eigenvalue of A iff $(A - \lambda I)\vec{x} = \vec{0}$ has non trivial solutions
- $\vec{x}$ is an eigenvector of A corresponding to λ iff $\vec{x}$ is a non trivial solution to $(A - \lambda I)\vec{x} = \vec{0}$

<u>Def</u> Let λ be an eigenvalue of A. The <u>eigenspace</u> of A corresponding to λ is $\mathrm{Nul}(A - \lambda I)$. ie the set containing $\vec{0}$ and all eigenvectors of A corresponding to λ.

Finding Eigenvalues

Tuesday, February 23, 2021 8:29 PM

<u>Theorem</u> The eigenvalues of a triangular matrix are its diagonal entries.

<u>Theorem</u> If $\vec{v_1} \dots \vec{v_r}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1 \dots \lambda_r$ of A, then $\{\vec{v_1} \dots \vec{v_r}\}$ is LI.

<u>Theorem</u> 0 is an eigenvalue of A iff A is not invertible.

Generally: solve the characteristic equation $\det(A - \lambda I) = 0$

Characteristic Equation

Tuesday, February 23, 2021 9:17 PM

Def Let A be an $n \times n$ matrix. The characteristic equation of A is $\det(A - \lambda I) = 0$ The characteristic polynomial is $\det(A - \lambda I)$ & it is a polynomial in λ of degree n

If λ_0 is an eigenvalue of A, then the algebraic multiplicity of λ_0 is the largest integer such that $(\lambda - \lambda_0)^k$ divides the characteristic polynomial.

Ex
$$\det(A - \lambda I) = \lambda^2 + \lambda - 6$$
$$= (\lambda + 3)(\lambda - 2)$$
$$= (\lambda - (-3))^1 (\lambda - 2)^1$$

algebraic multiplicity:
of -3 is 1
of 2 is 1

$$\det(A - \lambda I) = (\lambda - 1)^1 (\lambda - 2)^2$$

algebraic multiplicity:
of 1 is 1
of 2 is 2

Similarity

Tuesday, February 23, 2021 9:26 PM

Def $A_{n\times n}$ is similar to $B_{n\times n}$ if there exists an invertible matrix P such that $P^{-1}AP = B$

Note If A is similar to B, then

$P^{-1}AP = B$ then $AP = PB$ then $A = PBP^{-1}$

A & B are similar & $A \mapsto P^{-1}AP$ is a similarity transformation

Theorem If A and B are similar matrices, then A & B have the same characteristic polynomial & hence the same eigen values w/ same algebraic multiplicities.

Diagonalization

Thursday, February 25, 2021 1:39 PM

Def. Let A be a square matrix. Then A is diagonalizable if A is similar to a diagonal matrix. That is, $A = PDP^{-1}$ for some invertible matrix P & diagonal matrix D. We say that P diagonalizes A

Diagonalization Theorem Let A be an $n \times n$ matrix. Then A is diagonalizeable iff A has n LI eigenvectors. In fact, $A = PDP^{-1}$ where D is diagonal iff columns of P are n LI eigen vectors of A & in this case the diagonal entries of D are eigenvalues of A corresponding respectively to the eigen vectors in P.

$A = PDP^{-1}$ where $P = [\vec{v}_1 | \cdots | \vec{v}_p]$ where $\vec{v}_1 \ldots \vec{v}_p$ are the LI eigenvectors

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$ are the corresponding eigenvalues

Properties of Diagonalizable

Thursday, February 25, 2021 2:07 PM

Theorem Let $\lambda_1 \dots \lambda_p$ be distinct eigenvalues of A & let $E_{\lambda_1} \dots E_{\lambda_p}$ be the corresponding eigenspaces.
If S_i is a LI subset of E_{λ_i}, then $S_1 \cup \dots \cup S_p$ is LI.

Theorem Let A be an $n \times n$ matrix. If A has n distinct eigenvalues, then A is diagonalizable.

Theorem Let A be an $n \times n$ matrix w/ distinct eigenvalues $\lambda_1, \dots \lambda_p$:

a) For all $i = 1 \dots p$, dim(eigenspace for λ_i) $\leq$ algebraic multiplicity of λ_i
Geometric multiplicity: dim(eigenspace for λ_i)

b) A is diagonalizable iff $\sum_{i=1}^{n}$ geometric multiplicity of $\lambda_i = n$
iff the characteristic polynomial splits into linear factors and the algebraic multiplicity = geometric multiplicity

c) If A is diagonalizable & β_i is a basis for the eigenspace for λ_i, then $\beta_1 \cup \dots \cup \beta_p$ is an eigenvector basis for $\mathbb{R}^n$

Use of Diagonalization

Thursday, February 25, 2021 2:22 PM

If A is diagonalizable, then for some invertible matrix P & diagonal matrix D, $A = PDP^{-1}$ Then:

$$A^k = (PDP^{-1})^k$$

$$= (PDP^{-1})(PDP^{-1})\dots(PDP^{-1})$$

$$= PDI \dots IDP^{-1}$$

$$= PD^kP^{-1}$$

where the power of a diagonal matrix is the power of each diagonal element.

Dot Product

Friday, February 26, 2021 4:22 PM

Def Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. The <u>dot product</u> of $\vec{u}$ & $\vec{v}$ is

$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$ which is scalar

The dot product is an example of an inner product.

Def The <u>length / norm</u> of $\vec{u}$ is:

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Properties Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ & c be a scalar. Then

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

d) $\vec{u} \cdot \vec{u} \geq 0$ & $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = \vec{0}$

e) $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

f) $\|c\vec{u}\| = |c| \cdot \|\vec{u}\|$

Unit Vector

Friday, February 26, 2021 5:12 PM

<u>Def</u> A <u>unit vector</u> is a vector of length 1.

We can find a unit vector going in the same direction as $\vec{v} \in \mathbb{R}^n$ and $\vec{v} \neq 0$

$$\text{unit } \vec{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

and this process is called <u>normalizing</u> vector $\vec{v}$

Distance

Friday, February 26, 2021 5:14 PM

Def. For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the distance between $\vec{u}$ & $\vec{v}$ is

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

Orthogonality

Friday, February 26, 2021 5:19 PM

Def $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

Pythagorean Theorem: $\vec{u}$ & $\vec{v}$ are orthogonal iff $\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2$

Orthogonal Complement

Friday, February 26, 2021 5:29 PM

<u>Def</u> Let W be a subspace of $\mathbb{R}^n$ & let $\vec{z} \in \mathbb{R}^n$. Then z is <u>orthogonal</u> to W if $\vec{z}$ is orthogonal to every vector in W.

The <u>orthogonal complement</u> of W is

$$W^\perp = \{ \vec{z} \in \mathbb{R}^n \mid \vec{z} \cdot \vec{w} = 0, \ \vec{w} \in W \}$$

<u>Properties</u> Let W be a subspace of $\mathbb{R}^n$. Then

a) $\vec{x} \in W^\perp$ iff $\vec{x}$ is orthogonal to every vector in a set that spans W.

b) $W^\perp$ is a subspace of $\mathbb{R}^n$

c) $(W^\perp)^\perp = W$

d) $W \cap W^\perp = \{\vec{0}\}$

<u>Theorem</u> For any matrix A,

1) $(\text{Row } A)^\perp = \text{Nul } A$ 2) $(\text{Col } A)^\perp = \text{Nul } A^T$

Inner Product, Inner Product Space

Def. Let V be a vector space. An <u>inner product</u> on V is a function that assigns to each pair of vectors $\vec{u}, \vec{v} \in V$ to a scalar denoted $\langle \vec{u}, \vec{v} \rangle$

It must satisfy the following axioms:

For all $\vec{u}, \vec{v}, \vec{w} \in V$ and any scalar c:

1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3) $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$
4) $\langle \vec{u}, \vec{u} \rangle \geq 0$ & $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$

V with an inner product $\langle , \rangle$ is an <u>inner product space</u>

Examples of Inner Product Spaces

Monday, March 1, 2021 3:55 PM

The following are examples of inner product spaces:

- $\mathbb{R}^n$ w/ dot product
- Fix $t_0 \dots t_n$ to real numbers. Take $\mathbb{P}_n$ w/ $\langle , \rangle$ defined by $\langle p, q \rangle = p(t_0) + q(t_0) \dots p(t_n)\, q(t_n)$
- $C[a,b] = \{$continous functions on $[a,b]\}$ w/ $\langle , \rangle$ defined by $\langle f, g \rangle = \int_a^b f(t) \cdot g(t)\, dt$

Properties of Inner Product

Monday, March 1, 2021 4:01 PM

Def. Let V be an inner product space w/ inner product $\langle , \rangle$. For $\vec{u}, \vec{v} \in V$:

1) The length / norm of $\vec{u}$ is $||\vec{u}|| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$
2) $\vec{u}$ is a unit vector if $||\vec{u}|| = 1$
3) The distance between $\vec{u}$ & $\vec{v}$ is $||\vec{u} - \vec{v}||$
4) $\vec{u}$ & $\vec{v}$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$
5) $||\vec{u}||^2 = \langle \vec{u}, \vec{u} \rangle$ & $||c\vec{u}|| = |c| \cdot ||\vec{u}||$

Cauchy-Schwarz Inequality

Monday, March 1, 2021 4:57 PM

Theorem (Cauchy-Schwarz Inequality)

If $\vec{u}, \vec{v} \in V$, then $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$

Theorem (Triangle Inequality)

If $\vec{u}, \vec{v} \in V$, then $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Orthogonal Sets

Wednesday, March 3, 2021 6:04 PM

<u>Def</u> A subset $\{\vec{u}_1 \dots \vec{u}_p\}$ of $\mathbb{R}^n$ is <u>orthogonal</u> if $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$

<u>Theorem</u> If $S = \{\vec{u}_1 \dots \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ then S is LI & a basis for span $\{S\}$.

<u>Def</u> Let W be a basis of $\mathbb{R}^n$. An <u>orthogonal basis</u> for W is a basis for W which is orthogonal

<u>Theorem</u> Let $\{\vec{u}_1, \dots \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then for each $\vec{y} \in W$, if $\vec{y} = c_1\vec{u}_1 + \dots c_p\vec{u}_p$ then

$$c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} = \frac{\vec{y} \cdot \vec{u}_j}{\|\vec{u}_j\|^2} \quad \text{for } j = 1 \dots p$$

Orthonormal

Wednesday, March 3, 2021 6:24 PM

Def A subset $\{\vec{u}_1 \dots \vec{u}_p\}$ of $\mathbb{R}^n$ is orthonormal if it is orthogonal & every vector is a unit vector

Let W be a subspace of $\mathbb{R}^n$. An orthonormal basis for W is a basis for W which is orthonormal.

Orthogonal Matrix

Wednesday, March 3, 2021 6:27 PM

<u>Def</u>. An <u>orthogonal matrix</u> is a square matrix such that $U^{-1} = U^T$, which means that $U^T U = I_n$

<u>Theorem</u> An $m \times n$ matrix U has orthonormal columns iff $U^T U = I_n$

<u>Theorem</u> Let U be an $m \times n$ matrix with orthonormal columns & let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

1) $\| U\vec{x} \| = \| \vec{x} \|$

2) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

3) $(U\vec{x}) \cdot (U\vec{y}) = 0$ iff $\vec{x} \cdot \vec{y} = \vec{0}$

Orthogonal Projection (Vector to Vector)

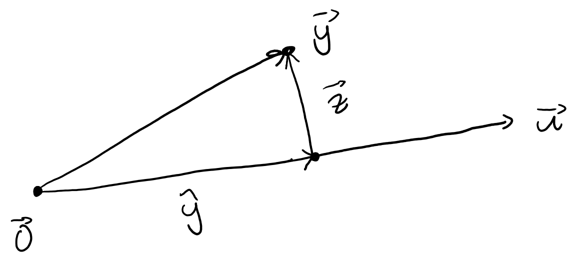
Wednesday, March 3, 2021 6:34 PM

Def Let $\vec{y}, \vec{u} \in \mathbb{R}^n$ where $\vec{u} \neq 0$. Let L be the line through $\vec{0}$ & $\vec{u}$. Then

orthogonal projection of $\vec{y}$ onto L is $\hat{y} = \text{proj}_L \vec{y} = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u}$

the component of $\vec{y}$ orthogonal to $\vec{u}$ is $\vec{z} = \vec{y} - \hat{y}$

Ex



Orthogonal Decomposition (Vector to Subspace)

Tuesday, March 9, 2021 10:38 PM

<u>Theorem</u> (Orthogonal Decomposition): Let W be a subspace of $\mathbb{R}^n$. Then each $\vec{y} \in \mathbb{R}^n$ can be written uniquely in the form $\vec{y} = \hat{y} + z$ where $\hat{y} \in W$ and $z = W^{\perp}$.

<u>Formula</u>: if $\{\vec{u}_1 \ldots \vec{u}_p\}$ is an orthogonal basis for W, then:

orthogonal projection of $\vec{y}$ onto W:

$$\text{proj}_W \vec{y} = \hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \cdot \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \cdot \vec{u}_2 + \ldots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \cdot \vec{u}_p$$

& component of $\vec{y}$ orthogonal to W:

$$z = \vec{y} - \text{proj}_W \vec{y} = \vec{y} - \hat{y}$$

Best Approximation Theorem

Tuesday, March 9, 2021 10:46 PM

<u>Theorem</u> Let W be a subspace of $\mathbb{R}^n$ & let $\vec{y} \in \mathbb{R}^n$. Then $\text{proj}_W \vec{y}$ is the closest point in W to $\vec{y}$. That is

$$\|\vec{y} - \text{proj}_W \vec{y}\| < \|\vec{y} - \vec{v}\| \quad \text{for any } \vec{v} \in W \text{ where } \vec{v} \neq \text{proj}_W \vec{y}.$$

We call $\text{proj}_W \vec{y}$ the <u>best approximation to $\vec{y}$ by elements of W</u>.

Projection onto Orthonormal Set

Tuesday, March 9, 2021 10:52 PM

__Theorem__ If $\{\vec{u_1} \dots \vec{u_p}\}$ is an __orthonormal__ basis for a subspace W of $\mathbb{R}^n$, then for all $\vec{y} \in \mathbb{R}^n$:

1) $\text{proj}_W \vec{y} = (\vec{y} \cdot \vec{u_1})\vec{u_1} + (\vec{y} \cdot \vec{u_2})\vec{u_2} + \dots + (\vec{y} \cdot \vec{u_p})\vec{u_p}$

2) If $U = [u_1 | \dots | u_p]$, then $\text{proj}_W \vec{y} = U U^T \vec{y}$.

Gram-Schmidt Process

Wednesday, March 10, 2021 4:56 PM

How to construct an orthogonal/orthonormal basis:

<u>Gram-Schmidt Process</u>: Let $\{\vec{x}_1, \ldots, \vec{x}_p\}$ be a basis for a nonzero subspace W of $\mathbb{R}^n$

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \sum_{i=1}^{p} \frac{\vec{x}_p \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \cdot \vec{v}_i$$

Then $\{\vec{v}_1 \ldots \vec{v}_p\}$ is an orthogonal basis for W &

$\{\text{norm } \vec{v}_1, \ldots, \text{norm } \vec{v}_p\}$ is an orthonormal basis for W.

Also $\text{span}\{\vec{x}_1, \ldots, \vec{x}_k\} = \text{span}\{\vec{v}_1 \ldots \vec{v}_k\}$ for $1 \le k \le p$

QR Decomposition

Wednesday, March 10, 2021 5:35 PM

<u>Theorem</u> Let A be an $m \times n$ matrix with LI columns, then A can be factored as $A = QR$ where

Q is an $m \times n$ matrix w/ orthonormal columns

R is an $n \times n$ upper triangular invertible matrix w/ positive diagonal entries

Let Q be the matrix whose columns are obtained by applying the orthonormal gram-schmidt process on the column vectors of A

Let $R = Q^T A$

Diagonalization of Symmetric Matrices

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Def. let A be a matrix. A is symmetric if $A^T = A$.

so $a_{ij} = a_{ji}$ for all i,j

so A is symmetric about its main diagonal

Ex $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is symmetric

Def. A is orthogonally diagonalizable if there exists an orthogonal matrix P & a diagonal matrix D such that $A = PDP^T = PDP^{-1}$

Theorem Let A be an $n \times n$ matrix. Then

A is orthogonally diagonalizable

iff A is symetric

iff A has orthonormal set of n eigenvectors.

Spectral Theorem for Symmetric Matrices

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<u>Thm</u>. Let A be an $n \times n$ symmetric matrix. Then

a) A has n real eigenvalues

b) Geometric multiplicities = algebraic multiplicities

c) Eigenspaces are mutually orthogonal

d) A must be orthogonally diagonalizable.

Singular Value Decomposition

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We can decompose an $m \times n$ matrix A:

$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V^T_{n \times n}$ where U, V are orthogonal